The effective energy of a lattice metamaterial

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Abstract

We study the sense in which the continuum limit of a broad class of discrete materials with periodic structures can be viewed as a nonlinear elastic material. While we are not the first to consider this question, our treatment is more general and more physical than those in the literature. Indeed, it applies to a broad class of systems including ones that possess mechanisms; and we discuss how the degeneracy that plagues prior work in this area can be avoided by penalizing change of orientation. A key motivation for this work is its relevance to mechanism-based mechanical metamaterials. Such systems often have “soft modes”, achieved in typical examples by modulating mechanisms. Our results permit the following more general definition of a soft mode: it is a macroscopic deformation whose effective energy vanishes – in other words, one whose spatially-averaged elastic energy tends to zero in the continuum limit.

1 Introduction

Homogenization was used to study large deformations of elastic composites nearly 40 years ago [7, 23], and discrete-to-continuous limits of nonlinear elastic structures have been a focus of attention for at least 20 years [1]. This paper has strong connections to both those threads, but its motivation comes from a much newer thread – namely the analysis of mechanism-based mechanical metamaterials. As we shall explain in section 1.1 by discussing some key examples, the systems we have in mind resemble porous elastic composites, but their essential properties can be captured by discrete lattice models. Besides their mechanisms, these systems often have *soft modes* – by which we mean macroscopic deformations that are not mechanisms, but that nevertheless have very little elastic energy. In the best-understood examples (such as the rotating squares metamaterial [12]), the soft modes are achieved by modulating a mechanism. It is natural to ask for a characterization of soft modes that doesn’t rely on a classification of the structure’s mechanisms. This question is important, because there are interesting examples for which we have no list or classification of the mechanisms (for example the Kagome metamaterial, which we discuss in some detail in sections 1.1 and 4, as well as other metamaterials with many mechanisms [6]). We believe that for lattice models of mechanism-based mechanical metamaterials, the soft modes are precisely the macroscopic deformations that minimize

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an appropriately-defined effective elastic energy. *The main goal of the present paper is to give sense to this assertion, by proving the existence of an effective elastic energy for a broad range of lattice models, including ones with mechanisms.* The assertion's consequences are explored and exploited in our paper [21], which focuses particularly on some conformal metamaterials, including the Kagome metamaterial and the rotating square metamaterial.

In the process of achieving the goal just enunciated, our paper also develops a new methodology. Indeed, we develop a new framework for the study of discrete-to-continuous limits of periodic structures, based on piecewise linear interpolation. Using our framework, arguments that are familiar for the study of continua have natural analogues for the study of discrete-to-continuous limits. While this framework is only used here to show the existence of an effective energy, we expect it to also have other applications.

Before discussing the paper's goals and accomplishments in more detail in section 1.2, let us elaborate further on the motivating issues discussed above.

1.1 Mechanism-based mechanical metamaterials

By definition, a mechanism of a mechanical system is a one-parameter family of deformations whose elastic energy is exactly zero, though they are not rigid motions. A broad variety of mechanism-based mechanical metamaterials have been considered in the literature (see, for example, [4]). To introduce the class that we shall study here it is convenient to begin with an example: a 2D *cut-out model* of the Kagome metamaterial. It is obtained by tiling the plane periodically by hexagons and triangles as shown in fig. 1(a) then cutting out the hexagons. The triangles that are left meet just at their vertices, which we view as hinges that can rotate without costing any elastic energy. This system has a one-parameter family of mechanisms with the same periodicity as the original structure, shown in fig. 1(b) for a particular value of the parameter. The mechanism deforms the holes left by cutting out the hexagons, but moves each triangle by a rigid motion; thus its elastic energy is exactly zero (which is the definition of a mechanism). We note that the mechanism changes the angles at which the triangles meet; this costs no elastic energy since we view the nodes as hinges.

This cut-out model of the Kagome metamaterial shows that our topic is closely related to the homogenization of nonlinear elastic composites; indeed, this model is more or less a porous elastic sheet. However, viewing the triangles as 2D nonlinearly elastic continua makes the model difficult to analyze. Fortunately, there is an alternative viewpoint which keeps the essential features of the problem and is more accessible to analysis. We therefore prefer the *spring model* of the Kagome metamaterial, obtained by treating the edges of the triangles as Hookean springs. We note that the mechanisms of the spring model are the same as those of the cut-out model, since triangles are rigid (in other words: if the vertices are moved in a way that leaves the length of each edge invariant then the deformation extends to a unique rigid motion of the entire triangle). We also note that while our springs are Hookean, the analysis of the spring model is a nonlinear problem, since we permit large deformations (and in particular large rotations); correspondingly, the elastic energy of the spring connecting x_i and x_j is its elastic constant times $\big(\frac{|u(x_i)-u(x_j)|}{|x_i-x_j|}-1\big)^2$, where $u(x_i)$ and $u(x_j)$ are the deformed positions of the nodes; as expected, this is not a quadratic function of the deformed positions.

In summary: considering the Kagome metamaterial as a periodic lattice of springs turns its analysis into a discrete problem (whose degrees of freedom are the deformations of the nodes), while keeping the essential features of this system (including the presence of mechanisms). Another favorite example – the rotating squares metamaterial (see e.g. [12, 15] and section 4.1.2) – admits a similar treatment: as a cut-out it is obtained by patterning the plane as a checkerboard then removing every white square, however its essential features are easily captured by a lattice of springs.[1] With these examples in mind, in the present work we shall focus on *lattice metamaterials*. While this class will be defined in section 2, we emphasize that it includes periodic lattices of springs.

As already mentioned earlier, a mechanical system with a mechanism will typically also have soft modes. To explain, let us continue our focus on the Kagome metamaterial. It has a soft mode taking the rectangular reference domain shown in fig. 2(a) to the sector of an annulus shown in fig. 2(b). Microscopically, the associated deformation uses the one-parameter family of mechanisms illustrated in fig. 1. Since the value of the parameter varies macroscopically, the deformation shown in fig. 2(b) is not a mechanism. However, the strain in each spring tends to zero in the limit as the ratio between the microscopic and macroscopic length scales tends to zero. As a result, the elastic energy of the soft mode is very small as one approaches the continuum limit.

It is natural to ask which macroscopic deformations can be accommodated by modulating the mechanism as in fig. 2(b). The answer lies beyond the scope of the present paper, but let us briefly discuss it anyway. To be realizable this way (with no overlapping of the triangles), the macroscopic deformation $u(x)$ should be a compressive conformal map (that is, $Du(x) = c(x)R(x)$ where $c(x)$ is scalar-valued and $R(x)$ takes values in $SO(2)$, with $1/2 < c(x) < 1$). Given such u, the process by which one gets an associated soft mode is discussed in [12]; an explanation why there are no other soft modes will be given in [21]. We note in passing that the Kagome metamaterial has *many* periodic mechanisms [19] (indeed, infinitely many [20]), and a soft mode can be obtained by modulating any of them. Thus the microscopic character of a soft mode is far from unique. This is illustrated by fig. 2(c), which achieves the same macroscopic deformation as fig. 2(b) by modulating a different periodic mechanism.

A body of literature has begun to develop concerning the mechanics of systems with a single one-parameter family of mechanisms; this includes the rotating squares metamaterial (the main focus of [12]) and a related but much broader family of kirigami-based examples (the focus of [25, 26]). These papers identify the soft modes of the systems they study, but they do much more. Indeed, to understand *which* soft mode will be achieved by a given loading condition, it is not enough to identify the class of all soft modes. Rather, one must minimize the leading-order elastic energy (plus the work done by the loads). For the cut-out or kirigami-based examples considered by these authors, this required modeling quantitatively the cost of modulation and the elastic energy due to the bending of thin necks that we treat as hinges in the present work. To connect those studies with the present paper: we have argued that the soft modes are macroscopic deformations whose effective energy vanishes, in other

[1] To model the rotating squares metamaterial by a lattice of springs, we can start with the square lattice then add extra diagonal springs to make some squares rigid; see for example Figure 6 in the supplementary information of [12] and this paper's section 4.1.2.

words whose spatially-averaged energy tends to zero in the limit as the separation of scales ε tends to zero. To predict the response of such a structure to loading, one should use the leading-order elastic energy (regardless of how it scales in ε). While the papers just discussed achieve such a goal for the specific systems they consider, their methods seem to require that there be a single one-parameter family of mechanisms. Thus, it remains an open question how something similar can be done for a system with many mechanisms like the Kagome metamaterial.

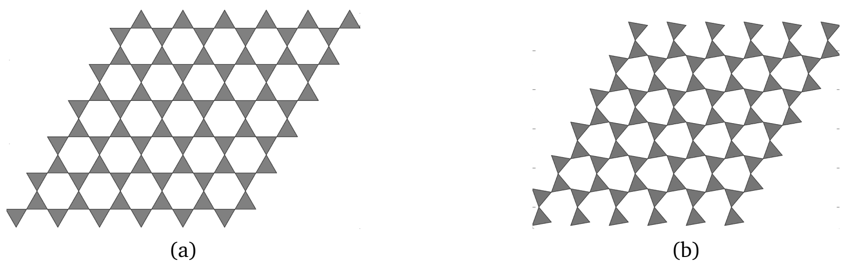

Figure 1: One of the periodic mechanisms on the Kagome lattice: (a) the reference state; (b) the deformed state.

Summarizing briefly the relevance of our work to mechanism-based mechanical metamaterials: while there has been impressive progress on systems with a single family of mechanisms, it is important to develop a framework that can also handle systems with many mechanisms (like the Kagome metamaterial). To get started, one requires an adequate definition of a soft mode. For periodic lattices of springs, we propose that the soft modes are the macroscopic deformations for which an appropriately-defined effective energy vanishes. Giving sense to and making use of this proposal requires

- understanding the existence and characterization of the effective energy;
- understanding, at least for some examples (such as the Kagome metamaterial), the macroscopic deformations where the effective energy vanishes.

This paper addresses the first bullet, while our forthcoming paper [21] addresses the second one.

We have not attempted to review the literature on mechanism-based mechanical metamaterials. There is, of course, a large body of work using *linear* elasticity to study lattices of springs. In that setting the discrete energy is convex, and periodic homogenization leads to an effective energy and even an effective Hooke's law (see e.g. [16] and [20]). However, there are lattices whose macroscopic behavior is not correctly described by linear elasticity (see e.g. [5]). The Kagome metamaterial is an example, since its linear elastic effective Hooke's law is nondegenerate, yet it has mechanisms achieving isotropic compression. To model small deformations of certain systems with mechanisms, Nassar et.al. [24] have proposed the use of Cosserat-type models. However, this approach requires knowing what mechanism is being activated, so it does not fully capture the relationship between soft modes and mechanisms. Our approach is entirely different; in particular, it makes no use of linearization, and it does not assume the existence (let alone a classification) of mechanisms.

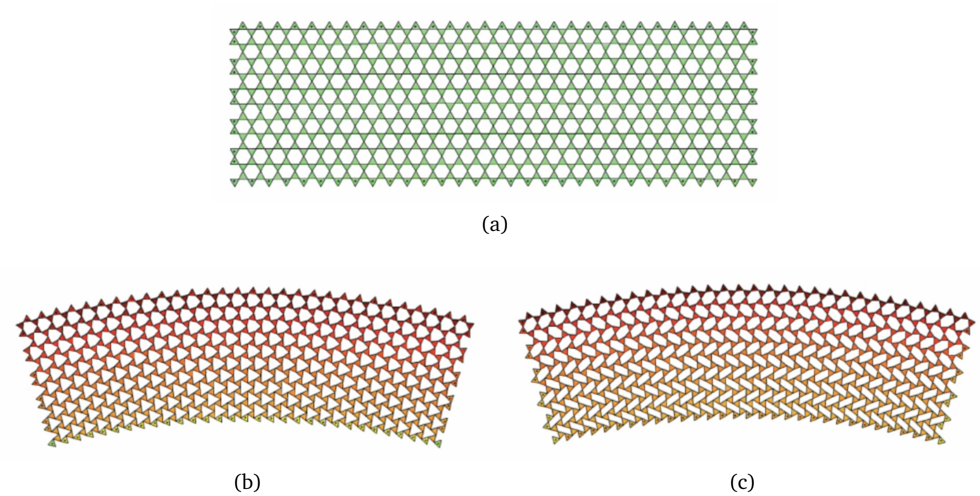

Figure 2: Soft modes in the Kagome lattice: (a) the reference state of the Kagome lattice; (b) a modulated version of the mechanism shown in fig. 1 (c) a modulated version of a different mechanism on the Kagome lattice. The color in each plot indicates the rotation angle of local equilateral triangles.

1.2 Overview of our framework and results

This paper's main goals are

(i) to discuss what it should mean for a lattice metamaterial to have an effective energy, and

(ii) to prove the existence of (and provide a characterization of) its effective energy.

Our approach to point (i) is rather different from previous work on lattices of springs (for example [1]). Indeed:

- our framework assumes periodicity but places virtually no other restriction on the geometry of the lattice; and
- it permits inclusion of a penalty for change of orientation, thereby avoiding the degeneracy that plagues prior treatments (without eliminating mechanisms).

Our approach to point (ii) is more familiar. Indeed, we adapt well-established tools from homogenization to the framework associated with (i), taking advantage of the analogy (already noted in section 1.1) between our problem and the analysis of spatially periodic nonlinearly elastic composites.

This section offers a non-technical overview. The story is long, since it involves modeling as well as analysis; therefore, we present it in modularized form.

THE MEANING OF AN EFFECTIVE ENERGY. In discussing the existence of an effective energy, we are considering a fixed macroscopic domain $\Omega \subset \mathbb{R}^N$ filled with an ϵ-scale version of our lattice metamaterial. Informally, we want to know whether in the limit $\epsilon \to 0$ we can view Ω as being filled with a nonlinear elastic material. This question has some subtlety, since an elastic deformation of a lattice is determined only at its nodes, and since our (geometrically nonlinear) elastic energies are expected to have local minima. To deal with the first issue, our mathematical definition of a lattice metamaterial will include the introduction of a (periodic) triangulation of $\mathbb{R}^N$ such that a deformation defined only at the nodes determines a unique piecewise linear function on all $\mathbb{R}^N$. To deal with the second issue, we use the framework of gamma-convergence. Informally, this means that the effective energy of a macroscopic deformation $u : \Omega \to \mathbb{R}^N$ is the smallest (limiting) energy achievable by deformations u^ϵ defined on the ϵ-scale lattice, when u^ϵ approximates the desired macroscopic deformation (in the sense that $\lim_{\epsilon\to 0} u^\epsilon = u$).

Since the effective energy captures the continuum limit of suitably-defined discrete energies, it is important to say a word about our discrete energies. While our framework is not limited to lattices of Hookean springs, it is convenient to focus for a moment on this special case. If x_i^ϵ and x_j^ϵ are nodes of the ϵ-scale lattice that are connected by a spring, then a deformation u^ϵ taking these points to $u^\epsilon(x_i)$ and $u^\epsilon(x_j)$ gives the spring the strain $\frac{|u^\epsilon(x_i^\epsilon)-u^\epsilon(x_j^\epsilon)|}{|x_i^\epsilon - x_j^\epsilon|} - 1$, and the spring's elastic energy is a constant times the square of the strain. For a lattice of springs, the discrete energy $E^\epsilon(u^\epsilon, \Omega)$ of the body Ω is, roughly speaking, the volume of Ω times the *spatial average* of the energies of all the springs in Ω. Thus, a macroscopic deformation with effective energy zero is one which can be approximately achieved on the ϵ-scaled lattice in such a way that the *average energy of all the springs tends to zero as* $\epsilon \to 0$. We emphasize that such a deformation need not be a mechanism, since the associated strains on the ϵ-scaled lattice need not be zero. For example, the deformations shown in figures 2(b) and (c) turn out to have strains of order ϵ in each spring; therefore the associated macroscopic deformation (which takes a rectangle to a sector of an annulus) has effective energy zero. However, mechanisms are still relevant; indeed, if there is a periodic mechanism[2] with macroscopic deformation gradient F, then the effective energy must vanish when $Du = F$.

USING SPRINGS ALONE IS NOT SUFFICIENT. It has long been understood that the effective energy of a lattice of springs can be very degenerate. To explain why, let us consider a simple example: the 2D lattice whose nodes are the integer points, with springs joining all nearest-neighbor and next-nearest-neighbor pairs.[3] Clearly, the folding deformation $u(x, y) = (-x, y)$ preserves the length of every spring, so its discrete energy is zero. Similarly, the lattice can be folded like an accordion (using folds along lines where x is an integer) to achieve any macroscopic compression in the horizontal direction. By symmetry, the same applies using folds where y is an integer. Thus, the effective energy of this system vanishes at $u(x, y) = (cx, y)$ and at $u(x, y) = (x, cy)$ for any $0 \le c \le 1$.[4] Evidently, the effective energy vanishes for deformations that are neither mechanisms nor soft modes, because a discrete energy that considers only the lengths of springs does not penalize change of orientation.

The resolution of this difficulty is simple: we must *inform the model that change of orientation is undesirable*. In the preceding example, each diagonal spring breaks a square into two triangles, and for

[2] By definition, a periodic mechanism has the form $u(x) = F \cdot x + \varphi(x)$, where F is a constant matrix and φ is a periodic function defined at nodes of the unit-scale lattice. In this case $u^\epsilon(x) = \epsilon u(x/\epsilon)$ is a mechanism of the ϵ-scaled lattice, which converges to the macroscopic deformation $u(x) = F \cdot x$ as $\epsilon \to 0$.

[3] Connecting only nearest neighbors would give a square lattice, which permits macroscopic shear with zero elastic energy. By introducing next-nearest-neighbor (diagonal) springs, one might expect at first to have a non-degenerate structure since the only orientation-preserving deformations with zero energy are rigid motions.

[4] In fact the effective energy also vanishes at $u(x, y) = (cx, dy)$ for any $0 \le c \le 1$ and $0 \le d \le 1$, for example by folding first along lines parallel to the x axis then along lines parallel to the y axis.

each such triangle T the nodal deformations determine an affine map u_T. Since $\det(Du_T)$ reveals the orientation of the map u_T, we can inform the model of our preference by adding a suitable function of such determinants to our energy. Moreover, if we use a function that vanishes when the determinant is positive, then the energy of an orientation-preserving deformation isn't being changed at all. This idea is not new; for example, a similar penalization is used (for a similar purpose) in [2].

Should one include such a penalization for *every* triangle? Not necessarily! As explained earlier, a key motivation for our work is the idea that the soft modes of a mechanical metamaterial are deformations for which a suitable-defined effective energy vanishes. The mechanical metamaterials that we have in mind have cut-out models as well as spring models, and the penalization should be applied only on regions that have not been cut out. In considering the Kagome metamaterial, for example, we must remember that the hexagonal regions are viewed as "holes;" therefore a penalization term should be included *only* for the equilateral triangles in fig. 1(a).

How large should the penalization be? This is a modeling choice, on which we need not take a definite position. But let us discuss what is at stake, focusing as usual on our idea that the soft modes of a lattice metamaterial should be the deformations where the effective energy vanishes. Based on spring energies alone, the minimum energy of a triangle is zero, but this minimum is achieved *both* when the triangle experiences an orientation-preserving rigid motion, *and* when it experiences an orientation-reversing rigid motion. Thus, in a deformation with small spring energy, the penalization will mainly be evaluated at determinants near ± 1. The penalization should of course be chosen so that it vanishes when the determinant is near 1 (so it doesn't interfere with any mechanisms), and it should be sufficiently large when the determinant is near -1. We expect that these are the *only* properties of the penalization that affect the zero-set of the effective energy.

Why use penalization, rather than simply prohibiting negative determinants? The answer is technical: while our theory accepts penalization, the proofs break down if we try to insist that our (microscopic) deformations have pointwise positive determinant. Indeed, our arguments rely on piecewise affine approximation, which raises the question whether a deformation with $\det(Du) > 0$ can be approximated by piecewise affine ones satisfying the same constraint. While an affirmative answer is known in two space dimensions [17, 18], this question is open in higher dimensions. Another issue involves interpolation: we need at certain points to interpolate between two deformations u and v which are known to be close in a weak norm (see appendix B). The obvious (and ultimately successful) idea is to use $u\varphi + v(1-\varphi)$ with a suitable choice of φ, however it does not seem possible to assure that such an interpolant has pointwise nonnegative determinant. Such issues are by now familiar in nonlinear elasticity; for a relatively recent discussion with additional references, see [11].

A GEOMETRY-INDEPENDENT, HOMOGENIZATION-LIKE FRAMEWORK. The systems that interest us have much in common with those studied by Alicandro and Cicalese in their seminal 2004 paper [1] concerning continuum limits of systems of springs. However, we cannot simply rely upon that work for the existence of an effective energy, because the geometries considered there are not sufficiently general. Indeed, when specialized to the periodic setting, [1] considers a periodic lattice of nonlinear springs *connecting the nodes of a square lattice*. By a linear change of variables, the analysis also applies to springs connecting the nodes of a Bravais lattice (in which, by definition, all nodes are translations of a single node by period vectors of the lattice). However, our spring model of the Kagome metamaterial does not have this character, since its nodes do not form a Bravais lattice. (In fact, its unit cell contains three nodes, as shown in fig. 3a.)

The methods used in [1] have been generalized to other settings, and we suppose they could be extended to our favorite examples (such as the Kagome lattice with a suitable penalization for change of orientation). In this paper, however, we pursue a different approach, which avoids any geometric hypothesis on the locations of the nodes. Instead, our framework emphasizes the hypothesis of periodicity and takes advantage of the analogy to homogenization of nonlinear elastic composites.

Our approach is developed in detail in section 2, but we outline it here. Since our structure is periodic, we consider a *unit cell* $U \subset \mathbb{R}^N$ – a parallelopiped containing the origin whose translates by vectors $v_1, \ldots, v_N$ tile all $\mathbb{R}^N$. The basic object we work with is the energy of the unit cell, $E(u, U)$, where u is a deformation (defined at nodes of the unscaled structure). The hypothesis of periodicity is captured by defining the energy of a translate of U by

$$E(u(x+\alpha), U+\alpha) = E(u(x), U)\,, \tag{1.1}$$

where we have introduced the convention that

$$\alpha = \alpha_1 v_1 + \cdots + \alpha_N v_N \quad \text{with } \alpha_i \in \mathbb{Z}\,. \tag{1.2}$$

The energy of the unit cell must be chosen so that the energy of the *entire* unscaled structure is

$$\sum_{\alpha_i \in \mathbb{Z}} E(u, U+\alpha)\,. \tag{1.3}$$

We hasten to add: the definition of $E(u, U)$ typically involves the values of u not only at the nodes in U, but also at some nodes in nearby translates of U; for example, in a network of springs, the springs that enter the definition of $E(u, U)$ may not lie entirely within U. In fact, the introduction of a unit cell is basically a bookkeeping device, which assures through (1.3) that adding the energies of U and its translates gets the total right.

To consider the effective energy, we must discuss the energy of the scaled structure. It is defined by elasticity scaling: if

$$\alpha = \alpha_1 v_1 + \cdots + \alpha_N v_N \quad \text{with } \alpha_i \in \epsilon\mathbb{Z} \tag{1.4}$$

and $u^\epsilon(x) = \epsilon u\left(\frac{x-\alpha}{\epsilon}\right)$ then

$$E^\epsilon(u^\epsilon, \epsilon U + \alpha) = E(u, U)\, \epsilon^N\,. \tag{1.5}$$

The scaled energy is again periodic, i.e. it satisfies an obvious analogue of (1.1). To explain the factor of ϵ^N on the right hand side of (1.5), we note that when u^ϵ is affine this definition makes $E^\epsilon(u^\epsilon, \epsilon U)$ proportional to the volume of ϵU.

The *energy of a domain* Ω filled by the scaled structure is, roughly speaking, the sum of the scaled energies of all translates of ϵU that lie inside Ω. But we must be careful, since when $\epsilon U + \alpha$ lies near

$\partial\Omega$ its energy could depend upon the values of u outside of Ω. This is a familiar issue in the area of discrete-to-continuum limits, and we resolve it in the usual way – by omitting the cells so close to $\partial\Omega$ that this is an issue. Thus, the energy of Ω using a deformation u^ϵ (defined at the nodes of the scaled structure) takes the form

$$E^\epsilon(u^\epsilon, \Omega) := \sum_{\alpha \in R_\epsilon(\Omega)} E^\epsilon(u^\epsilon, \epsilon U + \alpha), \tag{1.6}$$

where the definition of $R_\epsilon(\Omega)$ (given in section 2) is such that the sum excludes a boundary layer (whose width is over order ϵ) near $\partial\Omega$.

While we are mainly interested in lattices of springs (with penalization for change of orientation), our hypotheses upon the energy E are more abstract. Besides periodicity (discussed above), they are:

(a) nonnegativity,

(b) translation invariance, and

(c) upper and lower bounds relating E to the L^2 norm of ∇u on either U (for the lower bound) or the union of U and finitely many nearby cells (for the upper bound).

We refer to (2.12)–(2.15) for precise versions of these hypotheses, however we offer a few comments here. Concerning (a): there would be no essential difference if we only assumed that the energy was bounded below, since adding a constant would then achieve nonnegativity. Concerning (b): it is quite natural that u and $u + c$ should have the same energy when c is a translation (i.e. it takes the same value at every node), since mechanical structures are translation-invariant. Concerning (c): as we shall explain in section 2, we will identify a deformation u (which is defined only at the nodes of our structure) with a piecewise linear extension, so that ∇u makes sense.

While the examples discussed in this paper involve lattices of springs that rotate freely at nodes, our framework also permits favoring particular angles between the springs. For example, suppose x_1 is a node of the lattice that belongs to the unit cell U, and x_1 is joined by springs to nodes x_2 and x_3 (which may or may not belong to U). If we write $\ell_{ij} = u(x_j) - u(x_i)$ for the vector associated with the deformed spring from x_i to x_j, then adding

$$\Big| \ell_{12} \cdot \ell_{13} - |\ell_{12}||\ell_{13}| \cos\theta_0 \Big|$$

to the energy of the unit cell introduces a nonnegative term that vanishes only when the cosine of the angle between ℓ_{12} and ℓ_{13} is $\cos\theta_0$. Since this term is nonnegative with at most quadratic growth, adding it to an energy that already satisfies our lower bound will leave our framework intact[5].

[5] Sometimes, we also add torsional springs to penalize rotations at the hinges and to introduce resistance to bending. The resulting bending energy can be represented as a sum of terms of the form $k_s(\theta - \theta_0)^2$, where θ_0 is the preferred angle between two edges, and k_s is the torsional spring constant. For example, in the Kagome metamaterial, the bending energy can be characterized by summing $k_s(\theta - 2\pi/3)^2$ over all internal angles of the hexagonal holes. The torisional spring energy coincides with the energy proposed in the main text when the angle is close to θ_0, but deviates significantly as the angle moves is far away from θ_0.

Our framework *does not* require that $E(u, U)$ depend continuously on the deformation. Thus, for example, the term that penalizes change of orientation of a triangle T could have the form $|T| f^{\eta}(\det(\nabla u|_T))$ with

$$f^{\eta}(t) = \begin{cases} 1/\eta & \text{if } t \leq 0 \\ 0 & \text{if } t > 0 \end{cases} \tag{1.7}$$

where $\eta > 0$ is a small constant.

MAIN RESULTS AND METHODS. Our main result, theorem 2.11, asserts that in the limit $\epsilon \to 0$, the domain Ω can indeed be viewed as being occupied by a nonlinear elastic solid. In more technical terms: our discrete functionals $E^{\epsilon}(u^{\epsilon}, \Omega)$ gamma-converge to an effective energy of the form

$$E_{\text{eff}}(u, \Omega) = \int_{\Omega} \overline{W}(\nabla u)\, dx\,. \tag{1.8}$$

The integrand $\overline{W}$ does not depend on Ω; we view it, of course, as the hyperelastic energy density of the effective energy. The theorem also provides a variational characterization of $\overline{W}$, using which it is easy to see that $\overline{W} \geq 0$, and also that $\overline{W}$ is frame indifferent if the discrete energy has this property.

We remind the reader that to prove such a theorem we must provide, for any u,

(1) *an ansatz for the associated* u^{ϵ} – in other words, a family of discrete deformations u^{ϵ} (defined at the nodes of the ϵ-scaled structure, and converging in a suitable sense to u) such that $E^{\epsilon}(u^{\epsilon}, \Omega) \to E_{\text{eff}}(u, \Omega)$; and

(2) *a proof that this ansatz is asymptotically energetically optimal*, by showing that if any family of discrete deformations u^{ϵ} converges to u, then $\liminf_{\epsilon \to 0} E^{\epsilon}(u^{\epsilon}, \Omega) \geq E_{\text{eff}}(u, \Omega)$.

Theorem 2.11 considers our problem without boundary conditions. We also consider what happens when there is a Dirichlet-type boundary condition: Theorem 2.13 proves gamma-convergence in this setting as well. The limit is, as expected, the same effective energy (1.8) constrained by the Dirichlet boundary condition.

Our results are not a surprise, since they were already proved in [1] for the spring networks that satisfy that paper's hypotheses. However, our methods are quite different from those of [1]. Therefore our work provides an alternative perspective, even for problems where the results themselves are not new.

Our analysis is largely parallel to Müller's treatment of periodic homogenization problems in nonlinear elasticity [23]. It begins by addressing assertions (1) and (2) in the special case when u is affine; the variational characterization of $\overline{W}$ emerges from that argument. After treating the affine case, our analysis obtains assertion (1) for general u by considering piecewise linear functions then using a density argument. Our proof of assertion (2) for general u does not follow Müller; instead, it uses a blowup technique that was first applied to periodic homogenization by Braides, Maslennikov, and Sigalotti in [9].

We discussed earlier our view that for mechanism-based mechanical metamaterials, the soft modes are precisely the deformations for which a (suitably defined) effective energy vanishes. Given the form of the effective energy, this means that the soft modes are deformations u such that $\overline{W}(Du) = 0$ pointwise. It is therefore natural to ask: can we characterize, for specific examples, the zero-set of $\overline{W}$? The answer is yes: our forthcoming paper [21] shows that for our spring model of the Kagome metamaterial (with a suitable penalization for change of orientation), $\overline{W}(\lambda) = 0$ exactly when $\lambda = cR$ where $0 \leq c \leq 1$ and $R \in SO(2)$. Moreover, that paper's methods are not limited to Kagome; they also give a similar result for a spring-based model of the rotating squares metamaterial and other types of conformal metamaterials.

ORGANIZATION. The paper is structured as follows. Section 2 establishes our framework; this includes a careful treatment of our conditions on $E(u, U)$ and discussion about how some specific examples can be modeled this way. That section also gives precise statements of Theorems 2.11 and 2.13, as well as several lemmas concerning useful properties of the effective energy density $\overline{W}$. Section 3 provides the proofs of these results. Finally, section 4 illustrates the scope of our framework by discussing how the associated energy $E(u, U)$ should be chosen in some illustrative examples.

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2 Getting started

2.1 Lattice nodes, some basic examples, and U_n

To describe more precisely the N-dimensional lattice systems that interest us, we start by introducing some notation. As already indicated earlier, we start with a unit cell U (a parallelopiped containing the origin) and N vectors v_i such that the translates $U + \alpha$ tile[6] all $\mathbb{R}^N$ when $\alpha = \sum_{i=1}^{N} \alpha_i v_i$ with $\alpha_i \in \mathbb{Z}$. To identify the nodes of the lattice, we fix a basic set of nodes in the unit cell, $V = \{p_1, \ldots, p_{|V|}\} \subset U$; the full set $\mathcal{V}$ of nodes consists of all translates of elements of V:

$$\mathcal{V} = \bigcup_{\alpha_i \in \mathbb{Z}} (V + \alpha). \tag{2.1}$$

We assume that no two elements of V are lattice translates of one another, so each node of the lattice is *uniquely* expressible as $p + \alpha$ for some $p \in V$ and $\alpha = \sum \alpha_i v_i$.

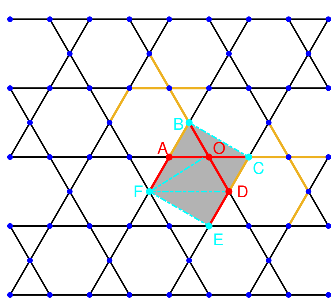
As an example, consider the 2D Kagome lattice shown in fig. 3a. A convenient choice of its unit cell U is the rectangle with vertices B, C, E, F, and a convenient choice of the basic set of nodes is $V = \{A, O, D\}$. If we choose the distance between two nearest nodes to be 1, then the translation vectors are $v_1 = (2, 0)^T$ and $v_2 = (1, \sqrt{3})^T$.

[6]The translated copies of the unit cell may have overlapping boundaries, but their interiors remain distinct and non-intersecting.

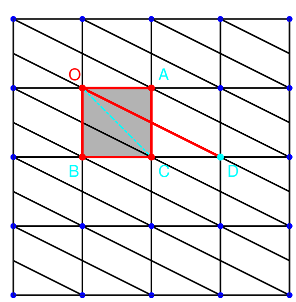
We want to endow such a lattice with an elastic energy. To do so, it is important to be clear about what we mean by an elastic displacement. We take the view that the displacement is an $\mathbb{R}^N$-valued function defined *only at nodes*. (Our situation is thus different from the theory of "reticulated structures," discussed e.g. in [10], where the displacements are defined on sets with nonzero volume.)

As already indicated in section 1.2, our elastic energy is determined by the energy of the unit cell $E(u,U)$, which we assume is nonnegative ($E \geq 0$) and translation-invariant ($E(u,U) = E(u+c,U)$ when c is a translation, i.e. it takes the same value at every node). As an example, consider our spring model of the Kagome metamaterial, with Hookean springs connecting each pair of nearest-neighbor nodes. If the unit cell is chosen as shown in fig. 3a, then it is convenient to let $E(u,U)$ be the energy of the six springs AO, BO, CO, DO, AF, DE, since each spring in the lattice is (uniquely) a translate of one of these. With this choice (and taking all the springs to be the same, and making a choice of spring constant) the energy of a translated unit cell $U+\alpha$ is

$$\begin{aligned} E(u,U+\alpha) = &\left(\Big|u(A+\alpha)-u(O+\alpha)\Big| - |A-O|\right)^2 + \left(\Big|u(B+\alpha)-u(O+\alpha)\Big| - |B-O|\right)^2 \\ &+ \left(\Big|u(C+\alpha)-u(O+\alpha)\Big| - |C-O|\right)^2 + \left(\Big|u(D+\alpha)-u(O+\alpha)\Big| - |D-O|\right)^2 \\ &+ \left(\Big|u(A+\alpha)-u(F+\alpha)\Big| - |A-F|\right)^2 + \left(\Big|u(D+\alpha)-u(E+\alpha)\Big| - |D-E|\right)^2. \end{aligned} \tag{2.2}$$



(a)



(b)

Figure 3: (a) The Kagome lattice: The shaded rectangle represents the unit cell U for the Kagome lattice, which contains three vertices A, O, D marked in red. These vertices can be translated to obtain the entire lattice. The solid red edges are those included in the energy calculation $E(u,U)$ in equation (2.2). A translated copy of these edges is marked in yellow to illustrate that all edges in the Kagome lattice can be viewed as translated copies of the red solid edges. The dotted lines indicate the triangular mesh used to interpolate the admissible deformations. (b) The square lattice with long-range interactions: the same coloring scheme is used to describe the unit cell, vertices within the unit cell, edges contributing to the energy $E(u,U)$, and the triangular mesh. The details are similar to those described for the Kagome lattice and are omitted for brevity.

The preceding example is somewhat unusual, because the definition of $E(u,U)$ uses only the values of u at nodes that belong to $\overline{U}$. Our framework does not require this, and such a choice is indeed impossible for many lattices of springs. As an example, consider the "square lattice with long-range interactions" shown in fig. 3b. The obvious unit cell U is a square with vertices O, A, B, C, and the natural way to define the elastic energy $E(u, U+\alpha)$ on the translated unit cell $U+\alpha$ is[7]

$$\begin{aligned} E(u, U+\alpha) = \frac{1}{2}\left(\Big|u(A+\alpha) - u(O+\alpha)\Big| - |A-O|\right)^2 + \frac{1}{2}\left(\Big|u(B+\alpha) - u(O+\alpha)\Big| - |B-O|\right)^2 \\ + \frac{1}{2}\left(\Big|u(B+\alpha) - u(C+\alpha)\Big| - |B-C|\right)^2 + \frac{1}{2}\left(\Big|u(A+\alpha) - u(C+\alpha)\Big| - |A-C|\right)^2 \\ + \left(\Big|u(O+\alpha) - u(D+\alpha)\Big| - |O-D|\right)^2. \end{aligned} \tag{2.3}$$

Evidently: the presence of long springs can require that $E(u, U)$ depend on displacements some distance from U.

This brings us to an important assumption which was omitted from the informal discussion in section 1.2: we shall assume that for some positive integer n,

$$E(u,U) \text{ depends only on the values of } u \text{ in the closure of the expanded cell}$$
$$U_n := \bigcup_{\alpha_i \in [-(n-1), n-1] \cap \mathbb{Z}} U + \alpha \tag{2.4}$$

with the usual convention $\alpha = \sum_{i=1}^N \alpha_i v_i$.

2.2 The piecewise linearization of a deformation, our basic energy bounds, and U_m

While an elastic displacement is characterized by its values at the nodes of the lattice, we want to also view it as a piecewise linear function defined on a suitable mesh. This is convenient because deformations of the scaled lattice can then be viewed as functions in a finite-dimensional subspace of H^1.

To this end, for any given lattice we fix – in addition to the structure introduced so far – a triangulation[8] of the unit cell U. The vertices of the triangulation must include the nodes of the lattice that lie in $\overline{U}$. We also permit the triangulation to use vertices that are not lattice nodes. At any non-lattice-node vertex y, we choose a way of writing y as a convex combination of finitely many lattice nodes (which might not belong to $\overline{U}$)

$$y = \sum_j \theta_j z_j \quad \text{where each } z_j \text{ is a lattice node, } 0 < \theta_j < 1, \text{ and } \textstyle\sum_j \theta_j = 1. \tag{2.5}$$

[7] The factor in the last row of eq. (2.3) is 1 because the edge OD lies entirely within the unit cell U, whereas the other edges are equally shared with neighboring cells.

[8] In dimension 3 or more, this would be a decomposition of U into simplices rather than triangles, however we shall use the term triangulation in any space dimension – a harmless abuse of language. Our "piecewise linear" functions are, of course, actually piecewise affine.

These non-lattice-node vertices are known as "ghost vertices". At each ghost vertex y, we take the value of the deformation to be

$$u(y) = \sum_j \theta_j u(z_j)\,. \tag{2.6}$$

This rule has the crucial property that it *preserves affine functions* – in other words, if $u(x) = \lambda \cdot x + c$ at the lattice nodes for some $N \times N$ matrix λ and some $c \in \mathbb{R}^N$, then $u(y) = \lambda \cdot y + c$ at every vertex of the triangulation, so the resulting piecewise linear function is pointwise equal to $u(x) = \lambda \cdot x + c$.

Given a piecewise linearization rule for the unit cell, we naturally obtain one for all $\mathbb{R}^N$ by periodic extension. It, too, preserves affine functions.

The piecewise linearizations of deformations play a fundamental role in our analysis. For one thing, they make it easy to discuss what it means for a family of deformations u^ϵ defined on ϵ-scaled versions of the lattice to converge as $\epsilon \to 0$ to a limit u, since the piecewise linearizations of u^ϵ are defined everywhere, not just at nodes. They also give sense to the upper and lower bounds that we require our unit cell energy $E(u, U)$ to satisfy, namely:

$$E(u, U) \le C_1 \Big(|\nabla u|^2_{L^2(U_n)} + |U_n| \Big) \tag{2.7}$$

and

$$E(u, U) \ge \max \Big\{ C_2 \Big(|\nabla u|^2_{L^2(U))} - D_2 |U| \Big), 0 \Big\} \tag{2.8}$$

for some positive constants C_1, C_2, and D_2. We discuss in section 4 how triangulations satisfying such estimates can be obtained for various 2D examples. Here, let us simply mention that for the 2D Kagome example with the unit cell shown in fig. 3a, the triangular mesh consisting of ΔAOB, ΔBOC, ΔCOD, $\Delta AOF, \Delta DOF, \Delta DEF$ is a convenient choice.

The upper bound eq. (2.7) – more precisely, the scaled version that we'll discuss presently – makes it natural that our gamma-limit be defined for u such that $\int_\Omega |\nabla u|^2 \, dx$ is finite. As the reader is by now well aware, we are especially interested in structures with mechanisms. The presence of a negative term $D_2|U|$ in the lower bound eq. (2.8) is crucial in that setting; indeed, $D_2|U|$ must clearly be larger than the maximum of $|\nabla u|^2_{L^2(U)}$ as u ranges over mechanisms (that is, over deformations such that $E(u, U) = 0$).

In using the L^2 norm of ∇u in the upper and lower bounds, we have made a choice. For a lattice of springs, conditions (2.7) and (2.8) place no constraint on the springs' character at small or even moderate strains, due to the terms involving $|U_n|$ and $|U|$ on the right hand side. However, these conditions require that the energies of the springs be of order (strain)^2 as strain $\to \infty$. While our arguments and results have natural analogues when the L^2 norms in (2.7) and (2.8) are replaced by L^p norms with $1 < p < \infty$, restricting our attention to $p = 2$ simplifies the discussion. Since it includes the case of Hookean springs – and since our motivation lies mainly in considering low-energy structures – this choice also seems quite natural from a mechanical viewpoint.

We note that on the right hand side of the bounds (2.7) and (2.8), the terms $|\nabla u|^2_{L^2(U_n)}$ and $|\nabla u|^2_{L^2(U)}$ refer to our piecewise linearization of u. If the triangulation has vertices that are not lattice nodes, then these terms may depend on values of the deformation at lattice nodes in nearby cells. We shall assume, however, that

the restriction of ∇u to U_n depends only on the values of u at lattice nodes in the closure of

$$U_m := \bigcup_{\alpha_i \in [-(m-1), m-1] \cap \mathbb{Z}} U + \alpha \,. \tag{2.9}$$

(Here n and U_n are defined by (2.4), and it is clear from the definition that $m \geq n$.) It will be convenient later to have a name for the largest distance between two points in U_m, so we define

$$d_m = \sup_{x,y \in U_m} |x - y| \,. \tag{2.10}$$

Remark 2.1. *It is worth noting that the upper bound in eq.* (2.7) *(or the scaled version in eq.* (2.14)*) for the unit cell energy depends implicitly on the value of m, due to the potential presence of ghost vertices. When ghost vertices are not needed to define the affine interpolation of the discrete deformations on the lattice, we have $m = n$.*

2.3 The scaled lattice, $E^\epsilon(u^\epsilon, \Omega)$, and our admissible deformations

We have already introduced the scaled lattice and the scaled energy in section 1.2. The nodes of the scaled lattice are

$$\mathcal{V}^\epsilon := \epsilon \mathcal{V} \,; \tag{2.11}$$

the translated unit cells of this lattice are $\epsilon U + \alpha$ with α as in (1.4); and the scaled energy $E^\epsilon(u^\epsilon, \epsilon U + \alpha)$ was defined via elasticity scaling in (1.5). As an example: for the Kagome lattice with U and $E(u, U)$ given by fig. 3a and eq. (2.2), if all the springs have length 1 in the unscaled setting then

$$\begin{aligned} E(u^\epsilon, \epsilon U + \alpha) = &\left(\left| u^\epsilon(\epsilon A + \alpha) - u^\epsilon(\epsilon O + \alpha) \right| - \epsilon \right)^2 + \left(\left| u^\epsilon(\epsilon B + \alpha) - u^\epsilon(\epsilon O + \alpha) \right| - \epsilon \right)^2 \\ &+ \left(\left| u^\epsilon(\epsilon C + \alpha) - u^\epsilon(\epsilon O + \alpha) \right| - \epsilon \right)^2 + \left(\left| u^\epsilon(\epsilon D + v_\alpha) - u^\epsilon(\epsilon O + \alpha) \right| - \epsilon \right)^2 \\ &+ \left(\left| u^\epsilon(\epsilon A + \alpha) - u^\epsilon(\epsilon F + \alpha) \right| - \epsilon \right)^2 + \left(\left| u^\epsilon(\epsilon D + \alpha) - u^\epsilon(\epsilon E + \alpha) \right| - \epsilon \right)^2 . \end{aligned}$$

Our unscaled upper and lower bounds (2.7) and (2.8) have scaled versions, of course. Their right hand sides involve the piecewise linearization of u^ϵ (determined by our unscaled piecewise linearization scheme and elasticity scaling, or equivalently by using (2.5)–(2.6) when y is a vertex of the scaled triangulation and $\{z_j\}$ are nodes of the scaled lattice).

While our conditions on the energy have already been discussed, it is convenient to collect them in one place. Since we'll mainly be using the scaled versions, we state those here:

(1) the energy on the ϵ-scale unit cell is periodic, i.e. we have

$$E^\epsilon(u^\epsilon(x+\alpha), \epsilon U + \alpha) = E^\epsilon(u^\epsilon, \epsilon U) \tag{2.12}$$

for any $\alpha = \sum_{i=1}^N \alpha_i v_i$ with $\alpha_i \in \epsilon\mathbb{Z}$;

(2) the energy on the ϵ-scale unit cell must be translation-invariant, in the sense that for any vector $c \in \mathbb{R}^N$, we have

$$E^\epsilon(u^\epsilon, \epsilon U + \alpha) = E^\epsilon(u^\epsilon + c, \epsilon U + \alpha)\,; \tag{2.13}$$

(3) an upper bound: there exists $C_1 > 0$ (independent of α and ϵ) such that

$$E^\epsilon(u^\epsilon, \epsilon U + \alpha) \leq C_1\Big(|\nabla u^\epsilon|^2_{L^2(\epsilon U_n + \alpha)} + |\epsilon U_n + \alpha|\Big) \tag{2.14}$$

where U_n is defined by (2.4); and

(4) a lower bound: there exist $C_2 > 0$ and $D_2 \geq 0$ (independent of α and ϵ) such that

$$E^\epsilon(u^\epsilon, \epsilon U + \alpha) \geq \max\Big\{C_2\Big(|\nabla u^\epsilon|^2_{L^2(\epsilon U + \alpha)} - D_2|\epsilon U + \alpha|\Big), 0\Big\}\,. \tag{2.15}$$

(Note that we have included positivity in eq. (2.15) rather than stating it as a separate condition. As already mentioned in section 1.2, our energy *need not* be a continuous function of the nodal deformations.)

Turning now to the energy of a domain: we offered a definition of $E^\epsilon(u^\epsilon, \Omega)$ in eq. (1.6), which we repeat here for the reader's convenience:

$$E^\epsilon(u^\epsilon, \Omega) := \sum_{\alpha \in R_\epsilon(\Omega)} E^\epsilon(u^\epsilon, \epsilon U + \alpha)\,;$$

however to make this precise we need to define the set over which the sum ranges. When considering the limiting energy of a fixed domain Ω, it is natural to focus on deformations that are defined at lattice nodes in Ω, in other words u^ϵ in

$$\mathcal{A}_\epsilon(\Omega) = \{u^\epsilon(x) \mid u^\epsilon(x) \text{ has values on } \mathcal{V}^\epsilon \cap \Omega\}\,. \tag{2.16}$$

Now recall from (2.9) that to be sure right hand sides of our unscaled energy bounds are fully determined, we need u to have values at lattice nodes in the closure of U_m. Scaling this statement, we see that the cells $\epsilon U + \alpha$ included in the definition of $E^\epsilon(u^\epsilon, \Omega)$ should have their closures contained in Ω. To enforce this, we define

$$R_\epsilon(\Omega) := \{\alpha = \sum_{i=1}^N \alpha_i v_i \ :\ \alpha_i \in \epsilon\mathbb{Z} \quad \text{and} \quad \epsilon U_m + \alpha \subset\subset \Omega\}\,, \tag{2.17}$$

using the usual convention that $A \subset\subset B$ means $\overline{A} \subset B$.

We note that Ω need not be an open set for $E^\epsilon(u^\epsilon, \Omega)$ to be well-defined, and no regularity is needed for $\partial\Omega$. While our gamma-convergence results (theorems 2.11 and 2.13) are restricted to Lipschitz domains, in the course of the proofs it will sometimes be convenient to consider the discrete energy of a domain whose boundary is not obviously Lipschitz.

The upper and lower bounds (2.14) – (2.15) tell us that the ϵ-scale problem $\min_{u^\epsilon \in \mathcal{A}_\epsilon(\Omega)} E^\epsilon(u^\epsilon, \Omega)$ is more or less a variational problem posed in a finite-dimensional subspace of $H^1(\Omega)$. In preparation for rigorous analysis, it is important to be clear about the class of admissible deformations. It is slightly different from $\mathcal{A}_\epsilon(\Omega)$, since we want to treat u^ϵ *both* as a function defined at nodes of the scaled lattice *and* as a piecewise linear function, though in a neighborhood of $\partial\Omega$ the piecewise linearization of a deformation may depend on its values at nodes outside Ω. While we usually use the same notation u^ϵ for both a deformation defined at lattice nodes and its piecewise linearization, for clarity we suspend this practice for the following definition.

Definition 2.2. An admissible deformation is a pair $(u^\epsilon, \tilde{u}^\epsilon)$ such that

(a) u^ϵ belongs to $\mathcal{A}_\epsilon(\Omega)$, i.e. it is a deformation defined at all nodes of the scaled lattice that lie in Ω;

(b) $\tilde{u}^\epsilon \in H^1(\Omega)$ is the restriction to Ω of a piecewise linear function obtained by applying our piecewise linearization scheme to some deformation defined at nodes of the scaled lattice;

(c) $\tilde{u}^\epsilon = u^\epsilon$ at all nodes of the scaled lattice that lie in Ω; moreover, $\tilde{u}^\epsilon$ agrees with our piecewise linearization of u^ϵ at all vertices of the triangulation where the piecewise linearization of u^ϵ is fully determined (i.e. where its value depends only on the deformation at nodes of the scaled lattice that lie in Ω).

In practice we will usually drop the tilde, writing u^ϵ instead of $\tilde{u}^\epsilon$. No confusion should result, since by (c) the two functions agree wherever they are both well-defined. This definition of the admissible deformations is convenient, because the piecewise-linear version of u^ϵ is now an element of the ϵ-independent space $H^1(\Omega)$. We note that $E^\epsilon(u^\epsilon, \Omega)$ is finite for any admissible deformation, and summing the upper bounds (2.14) for the relevant scaled cells gives

$$
\begin{aligned}
E^\epsilon(u^\epsilon, \Omega) &\le C_1 \Big(\sum_{\alpha \in R_\epsilon(\Omega)} |\nabla u^\epsilon|^2_{L^2(\epsilon U_n + \alpha)} + |\epsilon U_n + \alpha| \Big) \\
&\le C_1 (2n-1)^N \left(|\nabla u^\epsilon|^2_{L^2(\Omega)} + |\Omega| \right). \qquad (2.18)
\end{aligned}
$$

(The second line holds since each integral $|\nabla u^\epsilon|^2_{L^2(\epsilon U + \alpha)}$ over the cell $\epsilon U + \alpha$ can appear in the integral $|\nabla u^\epsilon|^2_{L^2(\epsilon U_n + \beta)}$ for some β at most $(2n-1)^N$ times.) Evidently, the energy $E^\epsilon(u^\epsilon, \Omega)$ stays uniformly bounded when $|\nabla u^\epsilon|_{L^2(\Omega)}$ stays uniformly bounded.

A similar calculation gives the following lemma, which will be used repeatedly.

Lemma 2.3. *Suppose a collection of scaled unit cells $\{\epsilon U + \alpha^{(j)}\}_{j=1}^P$, a deformation u^ϵ, a domain Ω, and a constant M have the properties that* *(a) the piecewise linear representative of u^ϵ has $|\nabla u^\epsilon| \leq M$ on $\epsilon U_n + \alpha^{(j)}$ for each $j = 1, \ldots, P$, and*

(b) each of the expanded cells $\epsilon U_n + \alpha^{(j)}$ is contained in Ω .

Then

$$\sum_{j=1}^{P} E^\epsilon(u^\epsilon, \epsilon U + \alpha^{(j)}) \leq C_1(2n-1)^N (M^2+1)|\Omega| \,.$$

Proof. It suffices to argue as we did for (2.18). □

We also note that, as a consequence of the lower bound (2.15), control of $E^\epsilon(u^\epsilon, \Omega)$ implies control of the L^2 norm of ∇u^ϵ in a slightly smaller domain. This too will be used repeatedly:

Lemma 2.4. *For any domain Ω and any admissible deformation $u^\epsilon \in \mathcal{A}_\epsilon(\Omega)$,*

$$C_2 \sum_{\alpha \in R_\epsilon(\Omega)} \int_{\epsilon U + \alpha} |\nabla u^\epsilon|^2 \, dx \leq E^\epsilon(u^\epsilon, \Omega) + C_2 D_2 |\Omega| \,. \tag{2.19}$$

Proof. The lower bound (2.15) implies that

$$C_2 \int_{\epsilon U + \alpha} |\nabla u^\epsilon|^2 \, dx \leq E^\epsilon(u^\epsilon, \epsilon U + \alpha) + C_2 D_2 |\epsilon U + \alpha| \,,$$

and our assertion follows by simply adding these inequalities over all $\alpha \in R_\epsilon(\Omega)$. □

2.4 Boundary conditions, and gluing deformations together

For a continous variational problem involving $\int_\Omega W(\nabla u)\, dx$, we can impose a Dirichlet boundary condition by specifying u at $\partial\Omega$. Moreover, given a partition of Ω into two subdomains, we can construct a test function by specifying u on each subdomain (using choices that agree at the partition boundary). Also, given two test functions u_1 and u_2, it can useful to interpolate between them by considering $\varphi u_1 + (1-\varphi)u_2$, where φ is smooth with $0 \leq \varphi \leq 1$. It is well-known that things are different in the context of discrete-to-continuous limits. Focusing on the framework of this paper, the issues are two-fold:

(a) For any scaled cell $\epsilon U + \alpha$, the associated energy $E^\epsilon(u^\epsilon, \epsilon U + \alpha)$ can depend on the values of u^ϵ at nodes of the scaled lattice in the larger set $\epsilon U_n + \alpha$. Moreover, our basic upper bound (2.14) involves the piecewise linearization of u^ϵ on U_n – which can depend on the values of u^ϵ in the still larger set $\epsilon U_m + \alpha$. Thus, our discrete problem is (a little bit) nonlocal.

(b) Our use of piecewise linearization introduces an additional issue. Consider, for example, the construction of a test function by interpolation, whereby $u^\epsilon = \varphi u_1^\epsilon + (1-\varphi)u_2^\epsilon$ at nodes of the scaled lattice. Alas, the piecewise linearization of u^ϵ is not given by this formula. Therefore rather than use product rule to calculate ∇u^ϵ, we must use information from the piecewise linearization scheme.

Point (a) is rather standard, and our way of dealing with it is rather standard as well. Point (b) is less standard; we shall deal with it using some basic ideas from numerical analysis.

We begin with a discussion of "Dirichlet boundary conditions." Given a domain $\Omega \subset \mathbb{R}^N$ and a Lipschitz continuous function $\psi : \partial\Omega \to \mathbb{R}^N$, how shall we impose in our discrete setting something similar to $u = \psi$ at $\partial\Omega$? Replacing u by $u - \psi$, it suffices to discuss the discrete analogue of $u = 0$ at $\partial\Omega$. Due to the nonlocality of the discrete problem, when working at scale ϵ we must require that u^ϵ *vanish in a layer* near $\partial\Omega$, whose thickness is of order ϵ:

Definition 2.5. For any domain Ω, let

$$\Omega_\epsilon = \left\{ x \in \Omega \,\middle|\, \text{dist}(x, \partial\Omega) > \epsilon d_m \right\}, \tag{2.20}$$

where d_m is defined by (2.10). We shall say that an admissible deformation u^ϵ "vanishes at $\partial\Omega$" if it belongs to

$$\mathcal{A}^0_\epsilon(\Omega) = \left\{ u^\epsilon \in \mathcal{A}_\epsilon(\Omega) \,\middle|\, u^\epsilon = 0 \text{ as a piecewise linear function on } \Omega \setminus \Omega_\epsilon \right\}. \tag{2.21}$$

The logic behind this definition is captured by the following two observations.

Remark 2.6. *If we extend $u^\epsilon \in \mathcal{A}^0_\epsilon(\Omega)$ by giving it the value 0 at nodes of the scaled lattice outside Ω, then the piecewise linearization of the extended deformation is the same as u^ϵ in the entire domain Ω. To see why, recall that for any scaled unit cell $\epsilon U + \alpha$, the piecewise linearization of u^ϵ in $\epsilon U_n + \alpha$ depends only on the values of u^ϵ at scaled lattice nodes in $\epsilon \overline{U}_m + \alpha$. Denoting the extended deformation by $\tilde{u}^\epsilon$, our claim is that the piecewise linearization of $\tilde{u}^\epsilon$ is equal to the piecewise linear function u^ϵ in the entire domain Ω. In fact:*

- *If $\epsilon \overline{U}_m + \alpha$ is contained in Ω then the piecewise linearizations of u^ϵ and $\tilde{u}^\epsilon$ in $\epsilon U_n + \alpha$ are fully determined by the values of u^ϵ at nodes of the scaled lattice in Ω. Thus, in this case $\tilde{u}^\epsilon = u^\epsilon$ in $\epsilon U_n + \alpha$, which is contained in Ω.*

- *If, on the other hand, $\epsilon \overline{U}_m + \alpha$ meets the complement of Ω, then (using the definition of d_m and Ω_ϵ), $\epsilon \overline{U}_m + \alpha$ lies in the exterior of Ω_ϵ. Since $u^\epsilon \in \mathcal{A}^0_\epsilon(\Omega)$, it vanishes at nodes of the scaled lattice that lie in $\Omega \setminus \Omega_\epsilon$. Thus, the extended deformation $\tilde{u}^\epsilon$ vanishes at all nodes of the scaled lattice that belong to $\epsilon \overline{U}_m + \alpha$. Since our piecewise linearization scheme preserves affine functions, the piecewise linearization of $\tilde{u}^\epsilon$ is identically zero in $\epsilon U_n + \alpha$. Using that the piecewise linear version of u^ϵ vanishes in $\Omega \setminus \Omega_\epsilon$, we conclude that u^ϵ and $\tilde{u}^\epsilon$ both vanish identically (as piecewise linear functions) on $(\epsilon U_n + \alpha) \cap \Omega$.*

As α varies, the sets $(\epsilon U_n + \alpha) \cap \Omega$ cover the entire set Ω; thus $\tilde{u}^\epsilon = u^\epsilon$ in all Ω, as asserted.

Remark 2.7. *Suppose Ω is partitioned into two subdomains Ω_1 and Ω_2, and suppose $u^\epsilon_i \in \mathcal{A}^0_\epsilon(\Omega_i)$ for*

$i = 1, 2$. *Then*

$$u^\epsilon(x) = \begin{cases} u_1^\epsilon(x) & x \in \Omega_1 \\ u_2^\epsilon(x) & x \in \Omega_2 \end{cases}$$

is an admissible deformation. When viewed as a piecewise linear function, $u^\epsilon(x) = 0$ *in the region* $dist(x, \partial\Omega_1 \cap \partial\Omega_2) \leq \epsilon d_m$*; in particular,* $u^\epsilon = 0$ *at all nodes* x *of the scaled lattice that lie in this region. This is consistent with our piecewise linearization scheme, since for any cell* $\epsilon U + \alpha$ *either*

(i) $(\epsilon\overline{U}_m + \alpha) \cap \Omega_1 = \emptyset$*, in which case the piecewise linearization of* u^ϵ *in* $\epsilon U_n + \alpha$ *is clearly* u_2^ϵ*; or*

(ii) $(\epsilon\overline{U}_m + \alpha) \cap \Omega_2 = \emptyset$*, in which case the piecewise linearization of* u^ϵ *in* $\epsilon U_n + \alpha$ *is clearly* u_1^ϵ*; or*

(iii) $(\epsilon\overline{U}_m + \alpha)$ *meets the common boundary* $\partial\Omega_1 \cap \partial\Omega_2$*, in which case* $u^\epsilon = 0$ *at all nodes of the scaled lattice in* $\epsilon\overline{U}_m + \alpha$*. Since our piecewise linearization scheme preserves affine functions, this is consistent (as expected) with* u^ϵ *being zero as a piecewise linear function in* $\epsilon U_n + \alpha$*.*

Turning now to a different issue: suppose φ is a continuous, piecewise linear function (with a macroscopic mesh that has nothing to do with our piecewise linearization scheme). What happens when we "discretize it" by taking $u^\epsilon = \varphi$ at nodes of the scaled lattice? The piecewise linearization of this u^ϵ is *not* everywhere equal to φ. Indeed, it agrees with φ at points which are far enough from a change in $\nabla\varphi$; but due to the nonlocality of our piecewise linearization scheme, it will be different from φ in an order-ϵ-thick layer around the set where $\nabla\varphi$ changes. To show that this layer has negligible effect on the total energy, we need an upper bound on ∇u^ϵ. This is a typical application of the following result, whose proof is given in appendix A:

Lemma 2.8. *For any Lipschitz continuous function* φ *and any cell* $\epsilon U + \alpha$ *of the scaled lattice, suppose* $u^\epsilon = \varphi$ *at all nodes of the scaled lattice that lie in* $\overline{U}_m + \alpha$*. Then the piecewise linearization of* u^ϵ *satisfies*

$$|u^\epsilon|_{L^\infty(\epsilon U_n + \alpha)} \leq |\varphi|_{L^\infty(\epsilon U_m + \alpha)}, \tag{2.22}$$

$$|\nabla u^\epsilon|_{L^\infty(\epsilon U_n + \alpha)} \leq C|\nabla\varphi|_{L^\infty(\epsilon U_m + \alpha)}, \text{ and} \tag{2.23}$$

$$|u^\epsilon - \varphi|_{L^\infty(\epsilon U_n + \alpha)} \leq C'\epsilon|\nabla\varphi|_{L^\infty(\epsilon U_m + \alpha)}. \tag{2.24}$$

The constants C *and* C' *in the latter two estimates depend on the details of our piecewise linearization scheme, but not on* ϵ *or* φ*.*

Finally we turn to a third issue, namely: estimating the piecewise linearization of $\varphi u_1^\epsilon + (1-\varphi)u_2^\epsilon$. (This issue arises in our version of an argument due to De Giorgi, which is briefly discussed near the end of the proof of lemma 3.1 then presented in full detail in appendix B.) The required estimate is provided by the following result:

Lemma 2.9. *For any Lipschitz continuous function* φ*, any cell* $\epsilon U + \alpha$ *of the scaled lattice, and any deformation* u^ϵ *that is defined at all nodes of the scaled lattice in* $\overline{U}_m + \alpha$*, suppose a deformation* h^ϵ *has*

$$h^\epsilon = \varphi u^\epsilon \quad \text{at nodes of the scaled lattice in } \epsilon\overline{U}_m + \alpha \quad .$$

Then the piecewise linearization of h^ϵ satisfies

$$|\nabla h^\epsilon|^2_{L^2(\epsilon U_n+\alpha)} \le C\Big(|u^\epsilon|^2_{L^2(\epsilon U_m+\alpha)}|\nabla\varphi|^2_{L^\infty(\epsilon U_m+\alpha)} + |\nabla u^\epsilon|^2_{L^2(\epsilon U_n+\alpha)}|\varphi|^2_{L^\infty(\epsilon U_n+\alpha)}\Big) \text{ and} \tag{2.25}$$

$$|h^\epsilon|^2_{L^2(\epsilon U_n+\alpha)} \le C'|u^\epsilon|^2_{L^2(\epsilon U_m+\alpha)}|\varphi|^2_{L^\infty(\epsilon U_m+\alpha)} \tag{2.26}$$

where the norms of u^ϵ on the right refer, as usual, to its piecewise linearization. The constants C and C' in this estimate depend only on the details of our piecewise linearization scheme; in particular, they do not depend on ϵ, φ, or u^ϵ.

The proof of lemma 2.9 is similar to (but more complicated than) that of lemma 2.8. It, too, is presented in appendix A.

2.5 Statements of our main results

Since our theorems use the notion of Γ-convergence, we start by defining what this means in the present context. Here and throughout the paper, the notation $u^\epsilon \rightharpoonup u$ means that $\{u^\epsilon\}$ remains uniformly bounded in $H^1(\Omega)$ and converges *weakly* to u.

Definition 2.10 (Γ-convergence)**.** We say that the family of discrete functionals $\{E^\epsilon(u^\epsilon,\Omega)\}$ Γ-converges to a functional $E_{\text{eff}}(u,\Omega)$ (with respect to the weak topology of $H^1(\Omega)$) if

(i) for every admissible sequence $\{u^\epsilon\}_{\epsilon>0}$ with $u^\epsilon \rightharpoonup u$ in $H^1(\Omega)$, we have

$$\liminf_{\epsilon\to 0} E^\epsilon(u^\epsilon,\Omega) \ge E_{\text{eff}}(u,\Omega)\,, \quad \text{and} \tag{2.27}$$

(ii) for every $u \in H^1(\Omega)$, there is an admissible sequence $\{u^\epsilon\}_{\epsilon>0}$ such that $u^\epsilon \rightharpoonup u$ in $H^1(\Omega)$ and

$$\lim_{\epsilon\to 0} E^\epsilon(u^\epsilon,\Omega) = E_{\text{eff}}(u,\Omega)\,. \tag{2.28}$$

It is easy to see that when this holds, the minimizers of $E_{\text{eff}}(u,\Omega)$ are precisely the weak limits of minimizing sequences of $E^\epsilon(u^\epsilon,\Omega)$. Thus, for a mechanism-based mechanical metamaterial we view the deformations with effective energy zero as soft modes, since they capture the macroscopic behavior of sequences u^ϵ for which $E^\epsilon(u^\epsilon,\Omega)$ tends to zero.

We should, perhaps, point out a limitation of this tool. If u is a deformation with effective energy zero, fulfilling part (ii) of the definition requires *only* supplying a sequence u^ϵ for which (2.28) holds. It does not require us to say, for example, that $E^\epsilon(u^\epsilon,\Omega)$ can be made of order ϵ^α for some $\alpha > 0$, let alone ask what the best such α might be.

Our main result is the following:

Theorem 2.11. *For any bounded, Lipschitz domain Ω the discrete energies $E^\epsilon(u^\epsilon,\Omega)$ Γ-converge in $H^1(\Omega)$ (with respect to the weak topology of $H^1(\Omega)$) to an effective energy of the form*

$$E_{\text{eff}}(u,\Omega) = \int_\Omega \overline{W}(\nabla u)\,dx\,. \tag{2.29}$$

Moreover, the effective energy density $\overline{W}(\lambda)$ *is independent of the domain* Ω*, and it has the following variational characterization:*

$$\overline{W}(\lambda) = \inf_{k\in\mathbb{N}} \inf_{\psi\in\mathcal{A}^0(kU)} \frac{1}{k^N|U|} \sum_{\alpha_1,\ldots,\alpha_N=0}^{k-1} E\left(\lambda x + \psi, U + \sum_{i=1}^{N} \alpha_i v_i\right), \tag{2.30}$$

where

$$kU = \bigcup_{\alpha_1,\ldots,\alpha_N=0}^{k-1} \left(U + \sum_{i=1}^{N} \alpha_i v_i\right) \tag{2.31}$$

and $\mathcal{A}^0(kU)$ *is the space of (unscaled) deformations of* kU *that "vanish at the boundary" in the sense of definition 2.5, in other words*

$$\mathcal{A}^0(kU) = \left\{ \begin{array}{l} \textit{admissible deformations } \psi \textit{ defined on } kU \textit{ whose piecewise} \\ \textit{linear representatives vanish when dist}(x, \partial(kU)) \leq d_m \end{array} \right\}.$$

(Note that this variational characterization uses the unscaled lattice, and that in (2.30), the expression $E(\lambda x + \psi, U + \sum_{i=1}^{N} \alpha_i v_i)$ *is our unscaled energy.)*

Remark 2.12. *A word is in order about the meaning of the (2.30), since* $E(u, U + \alpha)$ *depends on the values of* u *in* $\overline{U}_n + \alpha$*, and for some of the terms* $E(\lambda x + \psi, U + \alpha)$ *in (2.30) the set* $\overline{U}_n + \alpha$ *extends beyond* kU*. Since* $\psi \in \mathcal{A}^0(kU)$ *and recalling remark 2.6, we evaluate these terms by taking* $\psi = 0$ *outside* kU*.*

As the reader knows very well by now, we are especially interested in mechanical metamaterials. To explore the mechanical response of a metamaterial, it is natural to consider what happens when the deformation is specified at the boundary. This calls for an analogue of theorem 2.11 with a Dirichlet boundary condition. Of course, for the discrete problem at scale ϵ the "boundary condition" must be imposed in an order-ϵ-thickness layer near $\partial\Omega$: if ψ is an R^N-valued function defined near $\partial\Omega$, we say an admissible deformation "has boundary condition ψ" if it belongs to

$$\mathcal{A}_\epsilon^\psi(\Omega) = \left\{ u^\epsilon \in \mathcal{A}_\epsilon(\Omega) \,\middle|\, u^\epsilon - \psi \in \mathcal{A}_\epsilon^0(\Omega) \right\}. \tag{2.32}$$

This permits us to define the energy $E_\psi^\epsilon(u^\epsilon, \Omega)$ with Dirichlet boundary condition ψ:

$$E_\psi^\epsilon(u^\epsilon, \Omega) = \begin{cases} E^\epsilon(u^\epsilon, \Omega) & u^\epsilon \in \mathcal{A}_\epsilon^\psi(\Omega) \\ \infty & \text{otherwise}. \end{cases} \tag{2.33}$$

The following result shows that the effective energy $\int_\Omega \overline{W}(\nabla u)\, dx$ introduced in theorem 2.11 can also be used with a Dirichlet boundary condition.

Theorem 2.13. *For any bounded, Lipschitz domain* Ω *and any Lipschitz continuous boundary condition* $\psi : \partial\Omega \to \mathbb{R}^N$*, the discrete energies* $E_\psi^\epsilon(u^\epsilon, \Omega)$ Γ*-converge (with respect to the weak topology of* $H^1(\Omega)$*)*

to the effective energy

$$E^{\psi}_{\mathit{eff}}(u,\Omega) = \begin{cases} \int_\Omega \overline{W}(\nabla u)\,dx & u-\psi \in H^1_0(\Omega) \\ \infty & \text{otherwise}. \end{cases} \tag{2.34}$$

Before closing this section, we present three useful properties of the effective energy density $\overline{W}(\lambda)$.

Lemma 2.14. *The function $\overline{W}$ defined by (2.30) satisfies a quadratic growth condition: there exist constants $c_1, c_2, d_1 > 0$ such that*

$$\max\{c_1(|\lambda|^2 - d_1), 0\} \le \overline{W}(\lambda) \le c_2(|\lambda|^2 + 1) \tag{2.35}$$

for all $N \times N$ matrices λ.

Lemma 2.15. *$\overline{W}$ is Lipschitz continuous; in fact, there is a constant $c_3 > 0$ such that*

$$\left|\overline{W}(\lambda) - \overline{W}(\mu)\right| \le c_3(1 + |\lambda| + |\mu|)|\lambda - \mu| \tag{2.36}$$

for all $N \times N$ matrices λ and μ.

Lemma 2.16. *While the definition (2.30) of $\overline{W}(\lambda)$ uses test functions ψ with a Dirichlet boundary condition, the effective energy density also has an alternative characterization using periodic test functions. Indeed, let $\mathcal{A}^{\#}(kU)$ be the set of deformations defined at all nodes of our lattice that are k-periodic (that is, deformations ψ such that $\psi(x) = \psi(x + k\sum_{i=1}^N \alpha_i v_i)$ for $\alpha_1, \ldots, \alpha_N \in \mathbb{Z}$); and let $W^{\#}$ be the function obtained by replacing $\mathcal{A}^0(kU)$ by $\mathcal{A}^{\#}(kU)$ in the definition of $\overline{W}$:*

$$W^{\#}(\lambda) = \inf_{k\in\mathbb{N}} \inf_{\psi\in\mathcal{A}^{\#}(kU)} \frac{1}{k^N|U|} \sum_{\alpha_1,\ldots,\alpha_N=0}^{k-1} E(\lambda x + \psi, U + \sum_{i=1}^N \alpha_i v_i)\,. \tag{2.37}$$

Then in fact

$$\overline{W}(\lambda) = W^{\#}(\lambda)\,;$$

thus (2.37) gives an alternative variational characterization of $\overline{W}$.

The proofs of lemmas 2.14–2.16 are presented in section 3.2.

3 The proof of the main theorem

This section begins by establishing the assertions of theorem 2.11 in the special case when the macroscopic deformation is affine. Then, in section 3.2, we prove lemmas 2.14–2.16, which concern properties of the effective energy density $\overline{W}(\lambda)$. Besides being of interest in their own right, those properties are needed for the proofs of our theorems. With this groundwork complete, section 3.3 presents the proof of our main result, theorem 2.11, establishing Γ-convergence when no boundary condition is imposed. Finally, section 3.4 presents the proof of theorem 2.13, establishing Γ-convergence when a Dirichlet boundary condition is imposed.

3.1 The heart of the matter: affine limits

The proof of theorem 2.11 relies on the fact that every H^1 function is locally well-approximated by an affine function. Therefore, a crucial first step toward its proof lies in knowing that the theorem's assertions hold when the limit u is affine. Using the translation invariance of our energy (eq. (2.13)), it is sufficient to consider the case when u is linear.

Lemma 3.1. *For any $N \times N$ matrix λ, let $\overline{W}(\lambda)$ be defined by (2.30). Then for any bounded, Lipschitz domain Ω we have the following results:*

(a) For $u(x) = \lambda x$, there is a sequence of admissible deformations $\{u^\epsilon\}_{\epsilon>0}$ such that $u^\epsilon \rightharpoonup u$ in $H^1(\Omega)$ and

$$\lim_{\epsilon\to 0} E^\epsilon(u^\epsilon, \Omega) = |\Omega|\, \overline{W}(\lambda)\,. \tag{3.1}$$

Moreover, we can choose $u^\epsilon - \lambda x \in \mathcal{A}^0_\epsilon(\Omega)$, the set of functions that "vanish at $\partial\Omega$," defined by (2.21).

(b) If $u(x) = \lambda x$, then for any sequence of admissible deformations $\{u^\epsilon\}_{\epsilon>0}$ such that $u^\epsilon \rightharpoonup u$ in $H^1(\Omega)$ we have

$$\liminf_{\epsilon\to 0} E^\epsilon(u^\epsilon, \Omega) \geq |\Omega|\, \overline{W}(\lambda)\,. \tag{3.2}$$

Proof. Our argument follows the one Müller used for the continuous case in [23].

Proof of part (a) (finding a recovery sequence). We first assume that the double infimum in (2.30) is achieved for some specific $K \in \mathbb{N}$ and $\psi^K \in \mathcal{A}^0(KU)$, i.e.

$$\overline{W}(\lambda) = \frac{1}{K^N|U|} \sum_{\alpha_1,\ldots,\alpha_N=0}^{k-1} E(\lambda x + \psi, U + \sum_{i=1}^{N} \alpha_i v_i)\,. \tag{3.3}$$

While ψ^K is initially defined only on KU, it can be extended periodically to the entire lattice; the following argument uses this periodic extension, which (abusing notation slightly) we still call ψ^K. Tiling the plane with translates of ϵKU, we let $\widetilde{\Omega}_{K\epsilon}$ be the union of those tiles that are compactly contained in Ω_ϵ (the set defined by (2.20)). We claim that in this case

$$u^\epsilon(x) = \begin{cases} \lambda x + \epsilon\psi^K(\frac{x}{\epsilon}) & \text{for } x \in \widetilde{\Omega}_{K\epsilon} \\ \lambda x & \text{for } x \in \Omega \setminus \widetilde{\Omega}_{K\epsilon} \end{cases}$$

has the desired properties. Indeed, by remarks 2.6 and 2.7 this deformation is well-defined; moreover, it is easy to see that $u^\epsilon - \lambda x \in \mathcal{A}^0_\epsilon(\Omega)$ and $u^\epsilon \rightharpoonup \lambda x$ in $H^1(\Omega)$ as $\epsilon \to 0$. Our nontrivial task is to show that $\lim_{\epsilon\to 0} E^\epsilon(u^\epsilon,\Omega) = \overline{W}(\lambda)|\Omega|$. To this end, recall that $\widetilde{\Omega}_{K\epsilon}$ is a union of tiles that are translates of ϵKU. For any single tile, the sum of the energies of its cells is

$$\sum_{\alpha \text{ assoc one tile}} E^\epsilon(u^\epsilon, \epsilon U + \alpha) = \sum_{\alpha \text{ assoc one tile}} E^\epsilon(\lambda x + \epsilon\psi^K(x/\epsilon), \epsilon U + \alpha)$$
$$= \epsilon^N K^N |U| \overline{W}(\lambda) \tag{3.4}$$

using the definition of the scaled energy (1.5) together with our hypothesis (3.3). In doing this calculation, we have also used that $E^\epsilon(u^\epsilon, \epsilon U + \alpha)$ depends only the values of u^ϵ at nodes of the scaled lattice in $\epsilon\overline{U}_n + \alpha$, and that $\psi^K \in \mathcal{A}^0(KU)$; it follows by arguing as in remark 2.6 that the value of $E^\epsilon(\lambda x+\epsilon\psi^K(x/\epsilon), \epsilon U+\alpha)$ is oblivious to the fact that ψ^K is periodic, and is the same as if we extended it by 0 outside the given tile. Now, let us break the energy of u^ϵ in Ω into the part associated with the tiles and the rest:

$$E^\epsilon(u^\epsilon,\Omega) = \sum_{\substack{\alpha\,:\,\epsilon U+\alpha \text{ belongs}\\ \text{to a tile in } \widetilde{\Omega}_{K\epsilon}}} E^\epsilon(u^\epsilon, \epsilon U + \alpha) + \sum_{\substack{\alpha\in R_\epsilon(\Omega)\,:\,\epsilon U+\alpha\\ \text{is not in any tile}}} E^\epsilon(u^\epsilon, \epsilon U + \alpha)\,. \tag{3.5}$$

Applying (3.4) on each tile, we see that the first term equals $|\widetilde{\Omega}_{K\epsilon}|\overline{W}(\lambda)$. Since Ω is a Lipschitz domain, this converges to $|\Omega|\overline{W}(\lambda)$ as $\epsilon \to 0$.

We claim that the second term in (3.5) vanishes in the limit $\epsilon \to 0$. The key point is that for every cell $\epsilon U + \alpha$ counted in the second term we have $u^\epsilon = \lambda x$ on $\epsilon U_n + \alpha$ (this comes directly from the construction of u^ϵ). Moreover, these cells have the property that $\epsilon U_n + \alpha$ is contained in an order-ϵ width layer near $\partial\Omega$. Therefore lemma 2.3 shows that the second term of (3.5) is at most a constant times ϵ. Thus $\lim_{\epsilon\to 0} E^\epsilon(u^\epsilon,\Omega) = \overline{W}(\lambda)|\Omega|$, as desired.

Now we prove part (a) when the double infimum is not achieved; a diagonalization argument is needed in this case. We first fix $\delta > 0$ and choose $K \in \mathbb{N}$ and $\psi^\delta \in \mathcal{A}^0(KU)$ such that

$$\overline{W}(\lambda) \le \frac{1}{K^N|U|} \sum_{\alpha_1,\ldots,\alpha_N=0}^{K-1} E(\lambda x + \psi^\delta, U + \sum_{i=1}^N \alpha_i v_i) \le \overline{W}(\lambda) + \delta\,. \tag{3.6}$$

We obtain a lower bound on the L^2 norm of $\lambda + \nabla\psi^\delta$ by using the basic energy lower bound (2.15) on each cell and adding:

$$C_2 \int_{KU} |\lambda + \nabla\psi^\delta|^2\,dx \le (\overline{W}(\lambda) + \delta)|KU| + C_2 D_2 |KU|\,. \tag{3.7}$$

Now we proceed as above with ψ^δ in place of ψ^K – setting $u^{\epsilon,\delta}(x) = \lambda x + \epsilon\psi^\delta\left(\frac{x}{\epsilon}\right)$ on $\widetilde{\Omega}_{K\epsilon}$ and $u^{\epsilon,\delta}(x) = \lambda x$ on $\Omega \setminus \widetilde{\Omega}_{K\epsilon}$, where ψ^δ is the periodic extension of $\psi^\delta \in \mathcal{A}_0(KU)$ with period KU. The argument used for (3.7) is applicable on each scaled copy of KU in $\widetilde{\Omega}_{K\epsilon}$, and $\nabla u^{\epsilon,\delta} = \lambda$ in $\Omega \setminus \widetilde{\Omega}_{K\epsilon}$; these observations lead easily to a bound on $|\nabla u^{\epsilon,\delta}|_{L^2(\Omega)}$ that depends on λ but is independent of ϵ and δ as they tend to 0. Since the piecewise linear function $u^{\epsilon,\delta}$ equals λx at $\partial\Omega$, Poincare's inequality (applied to $u^{\epsilon,\delta}(x) - \lambda x$) gives a bound on $|u^{\epsilon,\delta}|_{L^2(\Omega)}$, so in fact we have uniform control on $u^{\epsilon,\delta}$ in

$H^1(\Omega)$.

Resuming now the stream of the earlier argument, it is clear that $u^{\epsilon,\delta} - \lambda x \in \mathcal{A}^0_\epsilon(\Omega)$ and $u^{\epsilon,\delta} \rightharpoonup \lambda x$ in $H^1(\Omega)$ as $\epsilon \to 0$ with δ held fixed. Since bounded sets in $H^1(\Omega)$ are compact in $L^2(\Omega)$, it follows that $\lim_{\epsilon\to 0} |u^{\epsilon,\delta} - \lambda x|_{L^2(\Omega)} = 0$. We also have

$$\lim_{\epsilon\to 0} E^\epsilon(u^{\epsilon,\delta}, \Omega) = \frac{|\Omega|}{K^N |U|} \sum_{\alpha_1,\ldots,\alpha_N=0}^{K-1} E(\lambda x + \psi^\delta, U + \sum_{i=1}^N \alpha_i v_i)$$

by arguing as we did earlier for ψ^K. Using (3.6), we deduce that

$$|\Omega|\overline{W}(\lambda) \le \lim_{\epsilon\to 0} E^\epsilon(u^{\epsilon,\delta}, \Omega) \le |\Omega|\big(\overline{W}(\lambda) + \delta\big)\,.$$

We now use a well-known diagonalization result, which is stated at the end of this subsection as lemma 3.2. Taking the function f in that lemma to be

$$f(\epsilon,\delta) = \big|E^\epsilon(u^{\epsilon,\delta}, \Omega) - |\Omega|\overline{W}(\lambda)\big| + \int_\Omega |u^{\epsilon,\delta}(x) - \lambda x|^2\,dx\,,$$

we obtain a sequence $u^{\epsilon,\delta(\epsilon)}$ converging strongly to λx in the L^2 norm such that

$$\lim_{\epsilon\to 0} E^\epsilon(u^{\epsilon,\delta(\epsilon)}, \Omega) = |\Omega|\overline{W}(\lambda)\,.$$

Since bounded subsets of $H^1(\Omega)$ are compact in the topology of weak convergence, we also have $u^{\epsilon,\delta(\epsilon)} \rightharpoonup \lambda x$, and the proof of part (a) is complete.

Proof of part (b) (the lower bound). This proof has three steps. In the first two, we assume that $u^\epsilon - \lambda x \in \mathcal{A}^0_\epsilon(\Omega)$, i.e. that $u^\epsilon - \lambda x$ "vanishes at $\partial\Omega$." The third step removes this restriction by using an argument due to De Giorgi. We shall present the first two steps here. Since the arguments used for the third step are very similar to those used in the setting of continuous periodic homogenization, this section provides just some references and a brief discussion about what is different in our setting. We do, however, provide the full details of this step in appendix B.

STEP 1: We consider the special case when $\Omega = \Omega_{\xi,s}$ is a scaled and translated version of U:

$$\Omega_{\xi,s} = \{x \,:\, x = \xi + sy \text{ for some } y \text{ in the interior of } U\} \tag{3.8}$$

where ξ is any vector in $\mathbb{R}^N$ and s is any positive real number. We shall show that for any sequence of admissible deformations satisfying $u^\epsilon - \lambda x \in \mathcal{A}^0_\epsilon(\Omega_{\xi,s})$, $\liminf_{\epsilon\to 0} E^\epsilon(u^\epsilon, \Omega_{\xi,s}) \ge |\Omega_{\xi,s}|\overline{W}(\lambda)$. (Weak convergence of u^ϵ to λx is not needed in this case.)

For any $\epsilon > 0$ we can choose a positive integer k and a translation $\alpha = \sum_{i=1}^N \alpha_i v_i$ with $\alpha_i \in \epsilon\mathbb{Z}$ such that

$$\Omega_{\xi,s} \subseteq \epsilon k U + \alpha \quad \text{where } kU \text{ is defined by (2.31)}$$

and such that the difference $(\epsilon k U + \alpha) \setminus \Omega_{\xi,s}$ has measure of order ϵ. Indeed, it suffices to choose the integer k so that $\epsilon < \epsilon k - s \leq 2\epsilon$. Since $\Omega_{\xi,s}$ is a parallelopiped whose edges are $\{sv_i\}_{i=1}^N$ while ϵkU is a parallelopiped whose edges are $\{\epsilon k v_i\}_{i=1}^N$, the condition $\epsilon k - s > \epsilon$ assures enough room to find the desired translation α while the condition $\epsilon k - s \leq 2\epsilon$ assures that the difference between the two sets has measure of order ϵ.

Let $\tilde{u}^\epsilon$ be the natural extension of u^ϵ to the larger domain $\alpha + \epsilon kU$:

$$\tilde{u}^\epsilon = \begin{cases} u^\epsilon & x \in \Omega_{\xi,s} \\ \lambda x & x \in (\epsilon kU + \alpha) \setminus \Omega_{\xi,s}\,, \end{cases}$$

which clearly has the property that $\tilde{u}^\epsilon - \lambda x \in \mathcal{A}^0_\epsilon(\epsilon kU + \alpha)$. The advantage of considering this extension is that we can estimate its energy using the definition of $\overline{W}$. Indeed, writing $\tilde{u}(x) = \epsilon^{-1}\tilde{u}^\epsilon(\epsilon x)$ and using the elasticity-scaling-based definition of the effective energy (1.5) we have

$$\sum_{\epsilon U + \beta \,\subseteq\, \epsilon kU + \alpha} E^\epsilon(\tilde{u}^\epsilon, \epsilon U + \beta) = \sum_{U + \gamma \,\subseteq\, kU + \epsilon^{-1}\alpha} \epsilon^N E(\tilde{u}, U + \gamma) \tag{3.9}$$

where on the left β is a translation preserving the scaled lattice while on the right γ is a translation preserving the unscaled lattice. (A word is in order about the meaning of (3.9). In a term on the left where $\epsilon U_n + \beta$ extends beyond $\epsilon kU + \alpha$ or a term on the right where $U_n + \gamma$ extends beyond $kU + \epsilon^{-1}\alpha$, we treat $\tilde{u}^\epsilon$ and $\tilde{u}$ as being equal to λx outside their respective domains. This is appropriate on account of remark 2.6, and it is consistent with remark 2.12 concerning the definition of $\overline{W}$.) Now using periodicity together with the definition of $\overline{W}$ and the fact that $\tilde{u} - \lambda x \in \mathcal{A}^0(kU + \epsilon^{-1}\alpha)$, we find that

$$\text{value of (3.9)} \geq \epsilon^N k^N |U| \overline{W}(\lambda) = |\epsilon kU| \overline{W}(\lambda)\,. \tag{3.10}$$

To obtain the desired conclusion that $\liminf_{\epsilon\to 0} E^\epsilon(u^\epsilon, \Omega_{\xi,s}) \geq |\Omega_{\xi,s}|\overline{W}(\lambda)$, we need only verify that

$$\lim_{\epsilon\to 0} |\epsilon kU| = |\Omega_{\xi,s}|\,, \tag{3.11}$$

$$\lim_{\epsilon\to 0} \Big| E^\epsilon(\tilde{u}^\epsilon, \epsilon kU + \alpha) - \sum_{\epsilon U + \beta \subseteq \epsilon kU + \alpha} E^\epsilon(\tilde{u}^\epsilon, \epsilon U + \beta) \Big| = 0\,, \quad \text{and} \tag{3.12}$$

$$\lim_{\epsilon\to 0} |E^\epsilon(\tilde{u}^\epsilon, \epsilon kU + \alpha) - E^\epsilon(u^\epsilon, \Omega_{\xi,s})| = 0\,. \tag{3.13}$$

The first assertion follows immediately from our choice of α and k, which were such that $\Omega_{\xi,s} \subseteq \epsilon kU + \alpha$ and $(\epsilon kU + \alpha) \setminus \Omega_{\xi,s}$ has measure of order ϵ. For the second assertion, we observe that the difference being estimated is

$$\sum_{\substack{\epsilon U + \beta \subseteq \epsilon kU + \alpha \\ \beta \notin R_\epsilon(\epsilon kU + \alpha)}} E^\epsilon(\tilde{u}^\epsilon, \epsilon U + \beta)\,. \tag{3.14}$$

In each of the cells $\epsilon U + \beta$ participating in this sum, $\tilde{u}^\epsilon = \lambda x$ throughout the expanded cell $\epsilon U_n + \beta$. Moreover, all these expanded cells lie within in an order-ϵ width layer near the boundary of $\epsilon kU + \alpha$, and the measure of this layer is of order ϵ. Therefore lemma 2.3 is applicable, and it bounds (3.14) by a constant times ϵ. Turning now to the third assertion: since $\Omega_{\xi,s} \subset \epsilon kU + \alpha$, we have $R_\epsilon(\Omega_{\xi,s}) \subset$

$R_\epsilon(\epsilon kU + \alpha)$, and the quantity to be estimated is

$$\sum_{\beta \in R_\epsilon(\epsilon kU+\alpha) \setminus R_\epsilon(\Omega_{\xi,s})} E^\epsilon(\tilde{u}^\epsilon, \epsilon U + \beta)\,. \tag{3.15}$$

Once again, for each cell $\epsilon U + \beta$ participating in this sum we have $\tilde{u}^\epsilon = \lambda x$ in the expanded cell $\epsilon U_n + \beta$; moreover, the expanded cells lie within an order-ϵ width layer about the boundary of $\epsilon kU + \alpha$. Therefore lemma 2.3 is again applicable, and it bounds (3.15) by a constant times ϵ. This establishes (3.13), completing Step 1.

STEP 2: Our goal in this step is the analogue of Step 1 with $\Omega_{\xi,s}$ replaced by any bounded, Lipschitz domain. Thus, we shall show for such Ω that for any sequence of admissible deformations satisfying $u^\epsilon - \lambda x \in \mathcal{A}^0_\epsilon(\Omega)$, $\liminf_{\epsilon\to 0} E^\epsilon(u^\epsilon, \Omega) \geq |\Omega|\overline{W}(\lambda)$. (As in Step 1, weak convergence of u^ϵ to λx is not needed for this argument.)

Since Ω is bounded, we can choose ξ and s such that $\overline{\Omega} \subset \Omega_{\xi,s}$. By part (a) of this Lemma there is a sequence v^ϵ with $v^\epsilon - \lambda x \in \mathcal{A}^0_\epsilon(\Omega_{\xi,s} \setminus \overline{\Omega})$ such that

$$\lim_{\epsilon\to 0} E^\epsilon(v^\epsilon, \Omega_{\xi,s} \setminus \overline{\Omega}) = |\Omega_{\xi,s} \setminus \overline{\Omega}|\, \overline{W}(\lambda)\,.$$

Combining v^ϵ and u^ϵ, we obtain a sequence of deformations $\tilde{u}^\epsilon$ defined on $\Omega_{\xi,s}$:

$$\tilde{u}^\epsilon(x) = \begin{cases} u^\epsilon(x) & x \in \Omega \\ v^\epsilon(x) & x \in \Omega_{\xi,s} \setminus \overline{\Omega}\,. \end{cases}$$

By remark 2.7, we have

$$E^\epsilon(\tilde{u}^\epsilon, \Omega_{\xi,s}) = E^\epsilon(u^\epsilon, \Omega) + E^\epsilon(v^\epsilon, \Omega_{\xi,s} \setminus \overline{\Omega}) + \sum_{\alpha\,:\,(\epsilon \overline{U}_m + \alpha) \cap \partial\Omega \neq \emptyset} E^\epsilon(\tilde{u}^\epsilon, \epsilon U + \alpha)\,. \tag{3.16}$$

The $\liminf$ of the left hand side is estimated by Step 1, and the limit of the second term on the right is known. We claim that the last term on the right is at most a constant times ϵ. Indeed, the cells $\epsilon U + \alpha$ that participate in the sum have the property that $\tilde{u}^\epsilon = \lambda x$ in $\epsilon \overline{U}_n + \alpha$, and these expanded cells lie in an order-ϵ width layer about $\partial\Omega$. Since Ω is a bounded, Lipschitz domain, the volume of that layer is of order ϵ. Thus lemma 2.3 applies, and it estimates the last term in (3.16) by a constant times ϵ. Using these observations, we deduce from (3.16) that

$$|\Omega_{\xi,s}|\overline{W}(\lambda) \leq \liminf_{\epsilon\to 0} E^\epsilon(u^\epsilon, \Omega) + |\Omega_{\xi,s} \setminus \overline{\Omega}|\, \overline{W}(\lambda)\,,$$

which leads immediately to the desired conclusion that $\liminf_{\epsilon\to 0} E^\epsilon(u^\epsilon, \Omega) \geq |\Omega|\overline{W}(\lambda)$.

STEP 3: We have thus far established the desired lower bound when "$u^\epsilon = \lambda x$ at $\partial\Omega$," even for a sequence that does not converge weakly to λx. The crucial third step is to prove the lower bound *without any boundary condition, provided that the sequence converges weakly to λx.* The technique for doing this relies on an argument of De Giorgi [14], and is by now well known. Müller provides full details in his paper [23] on the homogenization of nonlinear variational problems with periodic microstructure, and the discussion we offer here is parallel to his. There is, it seems, no known alternative to this argument; for example, Braides and Defranceschi use it in [9] (stating the required result as Lemma 2.2, and pointing to Section 11.1 of [8] for the proof). This technique has also been used in the discrete-to-continuous setting; in particular, Alicandro and Cicalese use appropriately adapted versions of it in Lemmas 3.7 and 3.8 of [1].

The main idea is to use "cutoff functions" φ^ϵ that are identically 1 in most of Ω but equal to 0 near $\partial\Omega$, and to consider $w^\epsilon(x) = u^\epsilon(x)\varphi^\epsilon(x) + (\lambda x)(1 - \varphi^\epsilon(x))$. The conclusion of Step 2 applies to w^ϵ; however, this is only useful if the energy of w^ϵ is asymptotically the same as that of u^ϵ. De Giorgi's argument demonstrates the existence of such φ^ϵ.

There is something different in our setting compared to that of continuous homogenization. Indeed, in the continuous setting one estimates ∇w^ϵ by simply using the product rule from calculus. In our setting, on the other hand, the relation $w^\epsilon(x) = u^\epsilon(x)\varphi^\epsilon(x) + (\lambda x)(1 - \varphi^\epsilon(x))$ can only be imposed at nodes of the lattice. Since ∇w^ϵ is the gradient of the *piecewise linearization* of this deformation, it cannot be estimated using the product rule; rather, one must use information about the piecwise linearization scheme. This is the character of lemma 2.9, which was stated in section 2.4 and is proved in appendix A.

Aside from the difference just noted, the arguments for Step 3 are rather familiar; therefore rather than present them here we have relegated them to appendix B. Combining those arguments with the conclusion of Step 2 completes the proof of part (b). □

Before closing this section, we state the diagonalization lemma that was used above for part (a) of lemma 3.1.

Lemma 3.2. *Let f be a function of two positive real numbers, taking values in the extended real line. Then there is a mapping $\epsilon \to \delta(\epsilon)$ such that $\epsilon \to 0$ implies $\delta(\epsilon) \to 0$ and*

$$\limsup_{\epsilon\to 0} f(\epsilon, \delta(\epsilon)) \le \limsup_{\delta\to 0}\limsup_{\epsilon\to 0} f(\epsilon, \delta)\,.$$

For a proof see for example Corollary 1.16 of [3].

Remark 3.3. *In our applications of lemma 3.2, ϵ and δ will often range over sequences approaching 0 rather than over all positive ϵ and δ near 0. The lemma is still applicable, by extending the discretely-defined function $f(\epsilon_j, \delta_k)$ to a suitable piecewise constant function $f(\epsilon, \delta)$ defined for positive ϵ and δ near 0. (Alternatively, the proof of the lemma can easily be repeated in the discrete setting.)*

3.2 Some easy properties of $\overline{W}(\lambda)$

We gave three useful properties of the effective energy density $\overline{W}$ at the end of section 2: (i) $\overline{W}(\lambda)$ satisfies a quadratic growth condition (lemma 2.14); (ii) $\overline{W}(\lambda)$ is Lipschitz continuous (lemma 2.15); and (iii) $\overline{W}(\lambda)$ has an alternative variational characterization using test functions with a periodic rather than affine boundary condition (lemma 2.16). We shall prove these lemmas in this subsection.

Proof of lemma 2.14. The upper bound for $\overline{W}(\lambda)$ is obtained by taking $k = 1$ and $\psi = 0$ in (2.30) and using lemma 2.3; this gives

$$\overline{W}(\lambda) \le \frac{1}{|U|} E(\lambda x, U) \le C_1 (2n-1)^N (|\lambda|^2 + 1)\,.$$

For the lower bound, it is obvious that $\overline{W}(\lambda) \ge 0$ since $E(u, U) \ge 0$ for every admissible deformation on the unit cell U. To show the other part of the lower bound, we use the convexity of function $\lambda \to |\lambda|^2$ to see that for every $k \in \mathbb{N}$ and $\psi \in \mathcal{A}_0(kU)$, the average energy of $\lambda x + \psi$ is lower bounded by

$$\begin{aligned}\frac{1}{k^N|U|} \sum_{\alpha_1,\ldots,\alpha_N=0}^{k-1} E(\lambda x + \psi, U + \sum_{i=1}^N \alpha_i v_i) &\ge \frac{C_2}{k^N|U|} \sum_{\alpha_1,\ldots,\alpha_N=0}^{k-1} \Big(|\lambda + \nabla\psi|^2_{L^2(U+\sum_{i=1}^N \alpha_i v_i)} - D_2|U| \Big) \\ &\ge \frac{C_2}{k^N|U|} \int_{kU} |\lambda + \nabla\psi|^2\, dx - C_2 D_2 \ge C_2\Big(|\lambda|^2 - D_2\Big)\,.\end{aligned}$$

In the last line we used Jensen's inequality, noting that since $\psi \in \mathcal{A}_0(kU)$, its piecewise linearization vanishes at the boundary (see remark 2.6), and therefore $\nabla\psi$ has integral zero. □

Proof of lemma 2.15. It suffices to show that $\overline{W}$ is rank-one convex, since rank-one convexity together with the quadratic growth condition $|\overline{W}(\lambda)| \le C(1 + |\lambda|^2)$ implies the desired result (2.36). (Indeed, rank-one convexity implies that $\overline{W}(\lambda)$ is separately convex as a function of the N^2 entries of the matrix λ; but separate convexity and the stated quadratic growth condition imply the desired result, see e.g. Proposition 2.32 in [13].)

The proof of rank-one convexity resembles the argument used to show that quasiconvexity implies rank-one convexity. Our goal is to that if $B - A$ has rank one and $0 < \theta < 1$ then

$$\overline{W}(\theta A + (1-\theta)B) \le \theta \overline{W}(A) + (1-\theta)\overline{W}(B)\,. \tag{3.17}$$

The proof is easiest to visualize when $B - A = a \otimes n$ with n parallel to one of the axes of $\mathbb{R}^N$, so let us focus for now on this case. Working on the domain $Q = (0,1)^N$, we shall use a test function that's piecewise linear except for a boundary layer near ∂Q, whose gradient takes the values A and B in layers orthogonal to n, with gradient A on approximately volume fraction θ and B on approximately volume fraction $1 - \theta$. Being more quantitative: for sufficiently small $\delta > 0$ our test function $\varphi^\delta : Q \to \mathbb{R}^N$ should be Lipschitz continuous such that

$$\begin{aligned} &\varphi^\delta(x) = \big(\theta A + (1-\theta)B\big)x && \text{for } x \text{ in a layer near } \partial Q \text{ and} \\ &|\nabla\varphi^\delta| \le c && \text{on } Q\,, \end{aligned}$$

where $c > 0$ is some constant (independent of δ). Moreover, Q should have a partition into regions

Ω_1, Ω_2, and $\Omega_3 = Q \setminus \overline{\Omega_1 \cup \Omega_2}$ such that

$$\begin{array}{cl} \Omega_1 & \text{is a union of finitely many rectangular layers where } \nabla\varphi^\delta = A, \\ \Omega_2 & \text{is a union of finitely many rectangular layers where } \nabla\varphi^\delta = B, \\ \overline{\Omega_1 \cup \Omega_2} & \text{forms a slightly smaller cube, omitting only a thin layer near } \partial Q, \end{array}$$

and

$$\Big| |\Omega_1| - \theta|Q| \Big| \le \delta|Q|, \qquad \Big| |\Omega_2| - (1-\theta)|Q| \Big| \le \delta|Q|, \qquad |\Omega_3| \le \delta|Q| \,. \tag{3.18}$$

The existence of such φ^δ is well-known; it is shown, for example, in Step 1 of the proof of Lemma 3.11 in [13].

We now use this framework to establish (3.17). Using part (a) of lemma 3.1, we choose a sequence of admissible deformations u^ϵ defined on Q such that:

(i) on each rectangular layer L in Ω_1 we have $u^\epsilon(x) - \varphi^\delta(x) \in \mathcal{A}^0_\epsilon(L)$ and $\lim_{\epsilon\to 0} E^\epsilon(u^\epsilon, L) = |L|\, \overline{W}(A)$;

(ii) on each rectangular layer L in Ω_2 we have $u^\epsilon(x) - \varphi^\delta(x) \in \mathcal{A}^0_\epsilon(L)$ and $\lim_{\epsilon\to 0} E^\epsilon(u^\epsilon, L) = |L|\, \overline{W}(B)$;

(iii) at all nodes of the scaled lattice outside $\Omega_1 \cup \Omega_2$ we take $u^\epsilon = \varphi^\delta$.

Since φ^δ is affine near ∂Q, we have (for any fixed δ)

$$u^\epsilon - \big(\theta A + (1-\theta)B\big)x \in \mathcal{A}^0_\epsilon(Q)$$

when ϵ is sufficiently small; therefore by Step 2 in the proof of lemma 3.1 part (b), we have

$$|Q|\, \overline{W}(\theta A + (1-\theta)B) \le \liminf_{\epsilon\to 0} E^\epsilon(u^\epsilon, Q) \,. \tag{3.19}$$

On the other hand, we claim that

$$E^\epsilon(u^\epsilon, Q) - \big(E^\epsilon(u^\epsilon, \Omega_1) + E^\epsilon(u^\epsilon, \Omega_2) + E^\epsilon(u^\epsilon, \Omega_3)\big) = O(\epsilon) \,. \tag{3.20}$$

Indeed, since Ω_1 is a union of finitely many disjoint layers L_j where $\nabla\varphi^\delta = A$, $\partial\Omega_1$ is the union of those layers' boundaries, so (using the definition (1.6)) we have

$$E^\epsilon(u^\epsilon, \Omega_1) = \sum_{\text{constituent layers } L_j} E^\epsilon(u^\epsilon, L_j)$$

when ϵ is sufficiently small. Similarly, $E^\epsilon(u^\epsilon, \Omega_2)$ is the sum of the energies of its constituent layers (where $\nabla\varphi^\delta = B$). Therefore for sufficiently small ϵ the left hand side of (3.20) is the sum of $E(u^\epsilon, \epsilon U + \alpha)$ as α ranges over $R_\epsilon(Q) \setminus \big(R_\epsilon(\Omega_1) \cup R_\epsilon(\Omega_2) \cup R_\epsilon(\Omega_3)\big)$. These scaled cells have the property that $\epsilon\overline{U}_m + \alpha$ meets $\partial\Omega_i$ for some i. For every such cell, we have $u^\epsilon = \varphi^\delta$ at the lattice nodes in $\epsilon U_m + \alpha$, so lemma 2.8 gives a uniform bound for $|\nabla u^\epsilon|_{L^\infty(\epsilon U_n + \alpha)}$. Moreover, for every such α, $\epsilon U_n + \alpha$ lies within an order-ϵ width layer near the boundary of Ω_i for some $i = 1, 2, 3$. Therefore lemma 2.3 shows that the cumulative energy of all these cells is of order ϵ (with an implicit constant that depends on δ, since the number of layers depends on δ). Combining (3.20) with properties (i) and (ii) of u^ϵ, we conclude that

$$\liminf_{\epsilon\to 0} E^\epsilon(u^\epsilon, Q) \leq |\Omega_1|\,\overline{W}(A) + |\Omega_2|\,\overline{W}(B) + \limsup_{\epsilon\to 0} E^\epsilon(u^\epsilon, \Omega_3)\,. \tag{3.21}$$

The last term on the right is at most a constant times $|\Omega_3|$, by another application of lemmas 2.3 and 2.8. Therefore (3.21) combines with (3.18) to give

$$\liminf_{\epsilon\to 0} E^\epsilon(u^\epsilon, Q) \leq |Q|\left(\theta\overline{W}(A) + (1-\theta)\overline{W}(B) + O(\delta)\right). \tag{3.22}$$

The desired conclusion (3.17) now follows by combining (3.19) with (3.22) then taking the limit $\delta \to 0$.

In the preceding argument, we restricted our attention to the case when $B - A = a \otimes n$ with n parallel to one of the coordinate axes, since in this case the construction of φ^δ is relatively simple and easily visualized. The general case is, however, almost the same: if $B - A = a \otimes n$ for any $a, n \in \mathbb{R}^N$, then an essentially identical argument can be used by taking Q to be a cube with sides parallel and perpendicular to n. Thus (3.17) holds whenever $B - A$ has rank one, and the proof is complete. □

Proof of lemma 2.16 (An equivalent variational form). It is obvious that $\overline{W}(\lambda) \geq W^\#(\lambda)$, since any $\psi \in \mathcal{A}^0(kU)$ has a natural periodic extension with period kU. We will use part (b) of lemma 3.1 to prove the opposite inequality. For any periodic ψ with periodicity kU, we consider a sequence of deformations of the form $v^\epsilon(x) = \lambda x + \epsilon\psi(\frac{x}{\epsilon})$ with $\epsilon \to 0$ chosen such that $1/(k\epsilon) \in \mathbb{N}$. Using the periodicity of ψ, we have

$$E^\epsilon(v^\epsilon, U) = \frac{1}{k^N}\sum_{\alpha_1,\ldots,\alpha_N=0}^{k-1} E(\lambda x + \psi, U + \sum_{i=1}^{N}\alpha_i v_i)\,.$$

But since ψ is periodic, v^ϵ converges weakly to λx in $H^1(\Omega)$, so we know from lemma 3.1 part (b) that

$$\liminf_{\epsilon\to 0} E^\epsilon(v^\epsilon, U) \geq |U|\,\overline{W}(\lambda)\,.$$

Combining these results gives

$$\frac{1}{k^N|U|}\sum_{\alpha_1,\ldots,\alpha_N=0}^{k-1} E(\lambda x + \psi, U + \sum_{i=1}^{N}\alpha_i v_i) \geq \overline{W}(\lambda)$$

for any $k \in \mathbb{N}$ and any $\psi \in \mathcal{A}^\#(kU)$. We deduce the desired conclusion that $W^\#(\lambda) \geq \overline{W}(\lambda)$ by minimizing over k and ψ. □

3.3 The proof of Theorem 2.11

We recall from definition 2.10 that proving Γ-convergence of $E^\epsilon(u^\epsilon, \Omega)$ to $\int_\Omega \overline{W}(\nabla u)\,dx$ requires showing two rather distinct results: the lower bound (2.27), and the existence of a "recovery sequence" (2.28). So far we have proved these assertions when u is affine. We turn now to the general case, when the limit can be any $u \in H^1(\Omega)$. As already mentioned in section 1.2, our methods are familiar from the literature on continuous homogenization problems: the recovery sequence is obtained using piecewise affine approximation, while the lower bound is proved by adapting the blowup argument of [9] to our discrete setting. Throughout this subsection Ω is assumed to be a bounded Lipschitz domain, since this is among the hypotheses of theorem 2.11.

Proof of theorem 2.11. We prefer to start with the recovery sequence, since the methods used for this are perhaps more familiar. To be clear: our goal in this part of the proof is to find, for any $u \in H^1(\Omega)$, a sequence $u^\epsilon \in \mathcal{A}^\epsilon(\Omega)$ such that $u^\epsilon \rightharpoonup u$ and $\lim_{\epsilon\to 0} E^\epsilon(u^\epsilon, \Omega) = E_{\text{eff}}(u, \Omega) = \int_\Omega \overline{W}(\nabla u)\, dx$.

Since Ω is a bounded, Lipschitz domain, the function u can be extended to a compactly supported H^1 function $\tilde{u}$ defined in all $\mathbb{R}^N$ with $|\tilde{u}|_{H^1(\mathbb{R}^N)} \le C_\Omega |u|_{H^1(\Omega)}$. The extension $\tilde{u}$ can be approximated by a smooth function u^η using mollification, and u^η can be approximated by a piecewise linear function u_δ using a mesh of order δ. (The mesh used to define u_δ has nothing to do with our lattice, nor with the scheme discussed in section 2.2 for determining the piecewise linearization of a deformation. In 2D, for example, the vertices of the mesh for u_δ could be the nodes of the square lattice with side length δ, if we triangulate each resulting square by introducing a diagonal edge.) We show in appendix C that by choosing the scale η of the mollification to depend appropriately on δ, we can arrange that

$$|\tilde{u} - u_\delta|_{H^1(\mathbb{R}^N)} \to 0 \text{ as } \delta \to 0 \quad \text{and} \tag{3.23}$$

$$|u_\delta|_{L^\infty(\mathbb{R}^N)} + |\nabla u_\delta|_{L^\infty(\mathbb{R}^N)} \le c_u \delta^{-a} \tag{3.24}$$

where c_u is a constant (depending on $|u|_{H^1(\Omega)}$) and a is a positive constant depending only on the spatial dimension N. We note that the functions u_δ are uniformly bounded (independent of δ) in $H^1(\mathbb{R}^N)$, since $|u_\delta|_{H^1} \le |u_\delta - \tilde{u}|_{H^1} + |\tilde{u}|_{H^1}$.

We shall obtain the recovery sequence by approximating u_δ with a suitable sequence of deformations $v^{\epsilon,\delta}$ defined on the ϵ-scaled lattice, then applying the diagonalization lemma 3.2. The argument shares many features with our proof of lemma 2.15. Some details are different, however, due to the negative exponent of δ in (3.24). Fortunately, that estimate will be needed only in an order-ϵ width boundary layer near the faces of the triangulation, so it leads to a term of order ϵ times a negative power of δ. Since our diagonalization lemma takes the limit $\epsilon \to 0$ before sending δ to 0, a term of this type is not problematic.

To define $v^{\epsilon,\delta}$ we apply part (a) of lemma 3.1 (combined with the translation invariance of our energy) to each of the simplices T in or near Ω on which u_δ is affine: $u_\delta|_T(x) = \lambda_T^\delta \cdot x + c_T^\delta$. The resulting $v_T^{\epsilon,\delta}$ has the following properties:

$$\lim_{\epsilon\to 0} \int_T |v_T^{\epsilon,\delta} - u_\delta|^2\, dx = 0\,, \tag{3.25}$$

$$v_T^{\epsilon,\delta}(x) - (\lambda_T^\delta \cdot x + c_T^\delta) \in \mathcal{A}_\epsilon^0(T)\,, \quad \text{and} \quad \lim_{\epsilon\to 0} E^\epsilon(v_T^{\epsilon,\delta}, T) = |T|\overline{W}(\lambda_T^\delta)\,. \tag{3.26}$$

Piecing these functions together, we define $v^{\epsilon,\delta}$ on the union of the simplices T that lie in or near Ω:

$$v^{\epsilon,\delta} = v_T^{\epsilon,\delta} \text{ at nodes of the } \epsilon\text{-scaled lattice that lie in } \overline{T}, \text{ if } \mathrm{dist}(T,\Omega) \leq 1\,. \tag{3.27}$$

This is well-defined, even when a node of the scaled lattice belongs to two or more simplices; to explain why, we observe that $v_T^{\epsilon,\delta} = u_\delta$ at ∂T, by the first part of (3.26) combined with remark 2.6. Our recovery sequence will be obtained by taking δ to be a suitable function of ϵ and restricting $v^{\epsilon,\delta(\epsilon)}$ to Ω.

The obvious idea is to apply our diagonalization lemma 3.2 with

$$f(\epsilon,\delta) = \left|E^\epsilon(v^{\epsilon,\delta},\Omega) - \int_\Omega \overline{W}(\nabla u)\right| + \int_\Omega |v^{\epsilon,\delta} - u|^2\,dx\,. \tag{3.28}$$

In the end we will make a slightly different choice – see (3.46) – but to explain the main ideas it is convenient to focus on (3.28). For the diagonalization lemma to be applicable, we need to show that

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\left|E^\epsilon(v^{\epsilon,\delta},\Omega) - \int_\Omega \overline{W}(\nabla u)\right| = 0 \tag{3.29}$$

and

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_\Omega |v^{\epsilon,\delta} - u|^2\,dx = 0\,. \tag{3.30}$$

Since $v^{\epsilon,\delta}$ has been defined simplex-by-simplex, it is convenient to work with inner and outer approximations of Ω that are unions of simplices on which u_δ is affine. For the inner approximation we choose

$$\Omega_{\text{in}}^\delta = \text{ union of all simplices } T \text{ such that } \overline{T} \subset \Omega, \tag{3.31}$$

while for the outer approximation we choose $\Omega_{\text{out}}^\delta$ such that

$$\Omega_{\text{out}}^\delta \text{ is a union of simplicies, it contains a } \delta\text{-neighborhood of } \Omega, \text{ and } \left|\Omega_{\text{out}}^\delta - \Omega\right| \to 0 \text{ as } \delta \to 0. \tag{3.32}$$

Since our simplices have diameter of order δ, it is obvious that $|\Omega - \Omega_{\text{in}}^\delta| \to 0$. So the volume of $\Omega_{\text{out}}^\delta \setminus \Omega_{\text{in}}^\delta$ tends to 0 as $\delta \to 0$, and therefore

$$\int_{\Omega_{\text{out}}^\delta\setminus\Omega_{\text{in}}^\delta} (1+|\nabla u_\delta|^2)\,dx \leq \int_{\Omega_{\text{out}}^\delta\setminus\Omega_{\text{in}}^\delta} (1+2|\nabla(u_\delta-\tilde{u})|^2 + 2|\nabla\tilde{u}|^2)\,dx \to 0 \quad \text{as } \delta\to 0. \tag{3.33}$$

To justify (3.29), our main task is to show that

$$\lim_{\epsilon\to 0} E^\epsilon(v^{\epsilon,\delta},\Omega_{\text{out}}^\delta) = \int_{\Omega_{\text{out}}^\delta} \overline{W}(\nabla u_\delta)\,dx \quad \text{and} \quad \lim_{\epsilon\to 0} E^\epsilon(v^{\epsilon,\delta},\Omega_{\text{in}}^\delta) = \int_{\Omega_{\text{in}}^\delta} \overline{W}(\nabla u_\delta)\,dx\,. \tag{3.34}$$

To explain why this implies (3.29), we first observe that since $\Omega_{\text{in}}^\delta \subset \Omega \subset \Omega_{\text{out}}^\delta$, the nonnegativity of our energy gives

$$E^\epsilon(v^{\epsilon,\delta},\Omega_{\text{in}}^\delta) \leq E^\epsilon(v^{\epsilon,\delta},\Omega) \leq E^\epsilon(v^{\epsilon,\delta},\Omega_{\text{out}}^\delta) \tag{3.35}$$

and the nonnegativity of $\overline{W}$ gives

$$\int_{\Omega_{in}^{\delta}} \overline{W}(\nabla u_\delta)\, dx \leq \int_{\Omega} \overline{W}(\nabla u_\delta)\, dx \leq \int_{\Omega_{out}^{\delta}} \overline{W}(\nabla u_\delta)\, dx\,. \tag{3.36}$$

On the other hand, we have $0 \leq \overline{W}(\nabla u_\delta) \leq c_2(1+|\nabla u_\delta|^2)$ from lemma 2.14, so (3.33) implies that

$$\int_{\Omega_{out}^{\delta}} \overline{W}(\nabla u_\delta)\, dx - \int_{\Omega_{in}^{\delta}} \overline{W}(\nabla u_\delta)\, dx \to 0 \tag{3.37}$$

as $\delta \to 0$. Finally, the Lipschitz property of $\overline{W}$ (lemma 2.15) gives

$$\left| \int_{\Omega} \overline{W}(\nabla u_\delta) - \overline{W}(\nabla u)\, dx \right| \leq c_3 \int_{\Omega} (1 + |\nabla u| + |\nabla u_\delta|)|\nabla u - \nabla u_\delta|\, dx \tag{3.38}$$

which tends to 0 as $\delta \to 0$ by Hölder's inequality. The desired conclusion (3.29) follows easily from (3.34) combined with (3.35)–(3.38).

We turn now to the proof of (3.34). It suffices to discuss the first assertion (concerning $\Omega^\delta_{\text{out}}$) since the justification of the second assertion (concerning $\Omega^\delta_{\text{in}}$) is entirely parallel. We recall from (3.26) that for each simplex $T \subset \Omega^\delta_{\text{out}}$ we have

$$\lim_{\epsilon \to 0} E^\epsilon(v_T^{\epsilon,\delta}, T) = \int_T \overline{W}(\nabla u_\delta)\, dx\,,$$

so we need to show that

$$E^\epsilon(v^{\epsilon,\delta}, \Omega^\delta_{\text{out}}) - \sum_{T \subset \Omega^\delta_{\text{out}}} E^\epsilon(v_T^{\epsilon,\delta}, T) \to 0 \quad \text{as } \epsilon \to 0. \tag{3.39}$$

Using the definition of the energy – (1.6) and (2.17) – the difference (3.39) is precisely

$$\sum_{\epsilon \overline{U}_m + \alpha \subset \Omega^\delta_{\text{out}}} E^\epsilon(v^{\epsilon,\delta}, \epsilon U + \alpha) \quad - \sum_{\substack{\epsilon \overline{U}_m + \alpha \subset T \text{ for} \\ \text{some simplex } T \subset \Omega^\delta_{\text{out}}}} E^\epsilon(v_T^{\epsilon,\delta}, \epsilon U + \alpha)\,.$$

The sum on the right is not changed if we replace $v_T^{\epsilon,\delta}$ by $v^{\epsilon,\delta}$. Indeed, $E^\epsilon(v_T^{\epsilon,\delta}, \epsilon U + \alpha)$ depends only on the values of $v_T^{\epsilon,\delta}$ at nodes of the scaled lattice in $\epsilon \overline{U}_n + \alpha$, by (2.4); and for the α that enter the sum, $\epsilon \overline{U}_n + \alpha \subset T$ (using that $m \geq n$). Therefore the difference (3.39) is equal to

$$\sum_{\substack{\epsilon \overline{U}_m + \alpha \text{ meets } \partial T \text{ for} \\ \text{some simplex } T \subset \Omega^\delta_{\text{out}}}} E^\epsilon(v^{\epsilon,\delta}, \epsilon U + \alpha)\,. \tag{3.40}$$

We come now to a key point: since $v_T^{\epsilon,\delta}(x) - (\lambda_T^\delta x - c_T^\delta) \in \mathcal{A}_0^\epsilon(T)$ by (3.26), for each α that participates in the preceding sum we have $v^{\epsilon,\delta}(x) = u_\delta(x)$ at all nodes of the scaled lattice in $\epsilon \overline{U}_m + \alpha$. Therefore lemma 2.8 combines with (3.24) to show that

$$\text{for each term in (3.40), } |\nabla v^{\epsilon,\delta}| \leq C\delta^{-a} \text{ in } \epsilon U_n + \alpha.$$

Since each scaled unit cell that participates in (3.40) lies in an order-ϵ neighborhood of ∂T for some simplex $T \subset \Omega^\delta_{\text{out}}$ – and since the number of simplices is of order $|\Omega|\delta^{-N}$ – we conclude from lemma 2.3 and the nonnegativity of our energy that

$$\text{the value of (3.40) is nonnegative, and bounded above by } C\epsilon$$

with a constant C that depends on δ, Ω, and u (but not ϵ). Taking the limit $\epsilon \to 0$, we obtain the desired conclusion (3.39).

We turn now to (3.30). Since u_δ approaches $\tilde{u}$ in L^2 as $\delta \to 0$ and $\Omega \subset \Omega^\delta_{\text{out}}$, it suffices to show that

$$\lim_{\epsilon\to 0} \int_{\Omega^\delta_{\text{out}}} |v^{\epsilon,\delta} - u_\delta|^2 \, dx = 0 \,. \tag{3.41}$$

The argument is similar to the proof of (3.34). We start with the obvious fact that

$$\int_{\Omega^\delta_{\text{out}}} |v^{\epsilon,\delta} - u_\delta|^2 \, dx = \sum_{T\subset\Omega^\delta_{\text{out}}} \int_{T_\epsilon} |v^{\epsilon,\delta} - u_\delta|^2 \, dx + \sum_{T\subset\Omega^\delta_{\text{out}}} \int_{T\setminus T_\epsilon} |v^{\epsilon,\delta} - u_\delta|^2 \, dx \tag{3.42}$$

where each sum is over all simplices $T \subset \Omega^\delta_{\text{out}}$, and (consistent with (2.21))

$$T_\epsilon = \Big\{ x \in T \,\Big|\, \text{dist}(x, \partial T) > \epsilon d_m \Big\} \,.$$

Observing that $v^{\epsilon,\delta}(x) = v^{\epsilon,\delta}_T(x)$ for $x \in T_\epsilon$, we have

$$\int_{T_\epsilon} |v^{\epsilon,\delta} - u_\delta|^2 \, dx = \int_{T_\epsilon} |v^{\epsilon,\delta}_T - u_\delta|^2 \, dx \to 0 \quad \text{as } \epsilon \to 0$$

using (3.25). It follows that the first term on the right side of (3.42) tends to 0 as $\epsilon \to 0$.

Preparing to estimate the other term, we claim that

$$|v^{\epsilon,\delta}(x)| \leq |u_\delta|_{L^\infty} \quad \text{when } x \in T \setminus T_\epsilon \quad . \tag{3.43}$$

Indeed, let $\epsilon U + \alpha$ be the scaled and translated unit cell that contains x. If $\epsilon \overline{U}_m + \alpha$ meets ∂T, then $\epsilon \overline{U}_m + \alpha$ cannot meet T'_ϵ for *any* simplex T', so we know from the first part of (3.26) that $v^{\epsilon,\delta} = u_\delta$ throughout $\epsilon \overline{U}_m + \alpha$, and lemma 2.8 provides the estimate (3.43) at x. If, on the other hand, $\epsilon \overline{U}_m + \alpha \subset T$ then $v^{\epsilon,\delta} = v^{\epsilon,\delta}_T$ in $\epsilon U + \alpha$; in particular, these two functions are equal at x. Since $x \in T \setminus T_\epsilon$ we know that $v^{\epsilon,\delta}_T(x) = u_\delta(x)$ from the first part of (3.26); therefore the estimate (3.43) is also valid in this case.

An estimate for the second term on the right side of (3.42) follows easily from (3.43) combined with our uniform bound (3.24) on u_δ. Remembering that $T \setminus T_\epsilon$ is an order-ϵ thick neighborhood of

∂T and that the total number of simplices is of order $|\Omega|\delta^{-N}$, we get that

the second term on the right side of (3.42) is at most $C\epsilon$

with a constant C that depends on δ, u, and Ω (but not ϵ). This converges to 0 as $\epsilon \to 0$, so the proof of (3.41) is complete.

We are still lacking one element. The preceding results let us conclude, using lemma 3.2, existence of $v^{\epsilon,\delta(\epsilon)}$ for which $E^\epsilon(v^{\epsilon,\delta(\epsilon)},\Omega) \to \int_\Omega \overline{W}(\nabla u)\,dx$ and $\int_\Omega |v^{\epsilon,\delta(\epsilon)} - u|^2\,dx \to 0$ as $\epsilon \to 0$. However, to know that $v^{\epsilon,\delta(\epsilon)} \rightharpoonup u$ in the weak topology on $H^1(\Omega)$ we need to know that $\nabla v^{\epsilon,\delta(\epsilon)}$ stays uniformly bounded in $L^2(\Omega)$ as $\epsilon \to 0$. The natural tool for proving this is our lower bound on the discrete energy, (2.15), which implies that

$$C_2 \int_{\epsilon U+\alpha} |\nabla v^{\epsilon,\delta(\epsilon)}|^2\,dx \le E^\epsilon(v^{\epsilon,\delta(\epsilon)}, \epsilon U+\alpha) + D_2|\epsilon U+\alpha|\,. \tag{3.44}$$

Summing these inequalities over all $\alpha \in R_\epsilon(\Omega)$ gives an upper bound for

$$\int_{\bigcup_{\alpha\in R_\epsilon(\Omega)}(\epsilon U+\alpha)} |\nabla v^{\epsilon,\delta(\epsilon)}|^2\,dx\,, \tag{3.45}$$

which misses an order-ϵ width layer near $\partial\Omega$.

We can fix this problem by changing the choice of $f(\epsilon,\delta)$ to which the diagonalization lemma is applied. Rather than the function $f_0(\epsilon,\delta)$ defined by (3.28), let us use

$$f_1(\epsilon,\delta) = f_0(\epsilon,\delta) + \frac{\epsilon}{\delta} + \left|E^\epsilon(v^{\epsilon,\delta},\Omega^\delta_{\text{out}}) - \int_{\Omega^\delta_{\text{out}}} \overline{W}(\nabla u_\delta)\right|. \tag{3.46}$$

The lemma is applicable, since we know using (3.34) that $\lim_{\delta\to 0}\lim_{\epsilon\to 0} f_1(\epsilon,\delta) = 0$. The resulting $v^{\epsilon,\delta(\epsilon)}$ has the property that $\epsilon/\delta(\epsilon) \to 0$ as $\epsilon \to 0$. Adding the estimates (3.44) over all $\alpha \in R_\epsilon\big(\Omega^{\delta(\epsilon)}_{\text{out}}\big)$ and writing

$$S_\epsilon = \bigcup_{\alpha\in R_\epsilon\left(\Omega^{\delta(\epsilon)}_{\text{out}}\right)} (\epsilon U+\alpha)$$

we get

$$C_2 \int_{S_\epsilon} |\nabla v^{\epsilon,\delta(\epsilon)}|^2\,dx \le E^\epsilon(v^{\epsilon,\delta(\epsilon)}, \Omega^{\delta(\epsilon)}_{\text{out}}) + D_2|\Omega^{\delta(\epsilon)}_{\text{out}}|\,. \tag{3.47}$$

To see that the left hand side of (3.47) controls $\int_\Omega |\nabla v^{\epsilon,\delta(\epsilon)}|^2\,dx$ when ϵ is sufficiently small, we recall that $\Omega^\delta_{\text{out}}$ contains a δ-neighborhood of Ω by (3.32); it follows that $\Omega \subset S_\epsilon$ when $\epsilon/\delta(\epsilon)$ is sufficiently small. To see that the right hand side of (3.47) stays bounded we observe that

$$\left|E^\epsilon(v^{\epsilon,\delta(\epsilon)}, \Omega^{\delta(\epsilon)}_{\text{out}}) - \int_{\Omega^{\delta(\epsilon)}_{\text{out}}} \overline{W}(\nabla u_{\delta(\epsilon)})\,dx\right| \to 0$$

since $f_1(\epsilon,\delta(\epsilon)) \to 0$; moreover $u_{\delta(\epsilon)}$ stays uniformly bounded in H^1 while we know from lemma 2.14

that $\overline{W}$ has quadratic growth. Thus the sequence

$$u^\epsilon = \text{restriction to } \Omega \text{ of } v^{\epsilon,\delta(\epsilon)}$$

converges weakly to u in $H^1(\Omega)$ and has $E^\epsilon(u^\epsilon, \Omega) \to \int_\Omega \overline{W}(\nabla u)\, dx$, as desired.

We turn now to the lower bound. Our task is to show that if $u^\epsilon \rightharpoonup u$ in $H^1(\Omega)$ then

$$\liminf_{\epsilon\to 0} E^\epsilon(u^\epsilon, \Omega) \geq \int_\Omega \overline{W}(\nabla u)\, dx\,. \tag{3.48}$$

Our proof relies on the fact that this has already been established when u is affine. We will localize the assertion (3.48) using a blow-up procedure. Since the blow-up of u at x_0 is its affine approximation, this procedure will permit us to deduce the desired result for any $u \in H^1(\Omega)$ from the one for affine limits. Since the argument is fairly long, we present it in several steps.

STEP 1: SETTING UP THE LOCALIZATION. We may (and do) focus on a subsequence $\epsilon_j \to 0$ such that

$$\lim_{j\to\infty} E^{\epsilon_j}(u^{\epsilon_j}, \Omega) = \liminf_{\epsilon\to 0} E^\epsilon(u^\epsilon, \Omega)\,. \tag{3.49}$$

We associate with this sequence a family of discrete nonnegative measures μ_j supported in Ω, by taking μ_j to have a point mass at each $\alpha \in R_\epsilon(\Omega)$ with weight $E^{\epsilon_j}(u^{\epsilon_j}, \epsilon_j U + \alpha)$; in other words

$$\mu_j(A) = \sum_{\alpha\in R_\epsilon(\Omega)} E^{\epsilon_j}(u^{\epsilon_j}, \epsilon_j U + \alpha)\, \delta_\alpha(A)\,, \qquad \delta_\alpha(A) = \begin{cases} 1 & \alpha \in A \\ 0 & \alpha \notin A\,. \end{cases}$$

Since $E^\epsilon(u^\epsilon, \Omega) = \sum_{\alpha\in R_\epsilon(\Omega)} E(u^\epsilon, \epsilon U + \alpha)$ we have

$$\mu_j(\Omega) = E^{\epsilon_j}(u^{\epsilon_j}, \Omega)\,. \tag{3.50}$$

Moreover, for any subset A of Ω we have

$$\mu_j(A) \geq E^{\epsilon_j}(u^{\epsilon_j}, A) \tag{3.51}$$

since $\epsilon_j \overline{U}_m + \alpha \subset A$ implies $\epsilon_j \overline{U}_m + \alpha \subset \Omega$. (In our applications of this inequality, the set A will be a small cube.) Passing to a further subsequence if necessary, we may suppose that the measures μ_j converge weakly to a limit μ. The weak limit is clearly nonnegative (since each μ_j is nonnegative) and it is supported on $\overline{\Omega}$, with

$$\mu(\overline{\Omega}) = \lim_{j\to\infty} \mu_j(\Omega) = \lim_{j\to\infty} E^{\epsilon_j}(u^{\epsilon_j}, \Omega)\,. \tag{3.52}$$

Taking the Radon-Nikodym decomposition of μ with respect to Lebesgue measure on $\mathbb{R}^N$, we have

$$\mu = \frac{d\mu}{dx}\mathcal{L}^N + \mu^s \tag{3.53}$$

where $\mathcal{L}^N$ is Lebesgue measure and $\mu^s \perp \mathcal{L}^N$. The singular part is nonnegative ($\mu^s \geq 0$) since $\mu(A) \geq 0$

for any measurable set A. Combining (3.52)-(3.53) and using the nonnegativity of μ^s we obtain

$$\lim_{j\to\infty} E^{\epsilon_j}(u^{\epsilon_j},\Omega) = \mu(\overline{\Omega}) \geq \mu(\Omega) \geq \int_\Omega \frac{d\mu}{dx}\, dx\,. \tag{3.54}$$

This framework reduces our task to proving that

$$\frac{d\mu}{dx}(x) \geq \overline{W}(\nabla u(x)) \quad \text{for Lebesgue-a.e. } x \in \Omega\,, \tag{3.55}$$

since the lower bound (3.48) then follows immediately using (3.49) and (3.54).

The rest of the proof is devoted to establishing (3.55). We shall prove it at $x = x_0$ when

(i) x_0 is a Lebesque point of μ, in other words

$$\frac{d\mu}{dx}(x_0) = \lim_{\rho\to 0} \frac{\mu\big(Q_\rho(x_0)\big)}{\rho^N} \tag{3.56}$$

where $Q_\rho(x_0)$ is an open cube centered at x_0 with side length ρ; and

(ii) x_0 is a Lebesgue point for u and ∇u, and moreover u is well-approximated near x_0 by its linear approximation in the sense that

$$\lim_{\rho\to 0} \frac{1}{\rho^2}\Big(\frac{1}{\rho^N}\int_{Q_\rho(x_0)} |u(x) - u(x_0) - \nabla u(x_0)\cdot(x-x_0)|^2\, dx\Big) = 0\,. \tag{3.57}$$

This suffices, since (3.56) holds Lebesgue-a.e. by a standard result from measure theory, and (3.57) holds Lebesgue-a.e. for any $u \in H^1(\Omega)$ (as a consequence, for example, of Theorem 3.4.2 in [27]).

STEP 2: BLOWING UP THE DISCRETE DEFORMATIONS. The deformation u^{ϵ_j} is defined at nodes of the ϵ_j-scaled lattice in Ω. Given $x_0 \in \Omega$ and sufficiently small $\rho > 0$, we want to consider the restriction of u^{ϵ_j} to a cube of size ρ around x_0, and to rescale it to a deformation w_j^ρ defined on the unit cube centered at 0 (which we denote by Q_1). By defining the rescaling appropriately, we will arrange that w_j^ρ be defined at the nodes of our $\frac{\epsilon_j}{\rho}$-scaled lattice that lie in Q_1.

Given x_0 and ϵ_j, there is a unique translation of the ϵ_j-scaled lattice that takes x_0 to the scaled unit cell $\epsilon_j U$:

$$x_0 = \epsilon_j(\xi_j + \alpha^j) \text{ where } \xi_j \in U \text{ and } \alpha^j = \textstyle\sum_{i=1}^N \alpha_i^j v_i \text{ with } \alpha_i^j \in \mathbb{Z} \text{ for each } i\,. \tag{3.58}$$

(Note that, contrary to our usual convention, α^j is a translation of the *unscaled* lattice rather than the scaled one. This is convenient because the following discussion involves two distinct scalings.) Our rescaled deformation is then

$$w_j^\rho(x) = \frac{u^{\epsilon_j}(x_0 - \epsilon_j\xi_j + \rho x) - u(x_0)}{\rho}\,. \tag{3.59}$$

This deformation is in $\mathcal{A}^{\epsilon_j/\rho}(Q_1)$ provided that $Q_\rho(x_0) - \epsilon_j\xi_j \subset \Omega$. Indeed, if x is a node of the

ϵ_j/ρ-scaled lattice, say

$$x = \tfrac{\epsilon_j}{\rho}(p_k + \beta) \text{ where } p_k \in V \text{ and } \beta = \textstyle\sum_{i=1}^N \beta_i v_i \text{ with } \beta_i \in \mathbb{Z} \text{ for each } i\,,$$

then u^{ϵ_j} is evaluated in (3.59) at

$$x_0 - \epsilon_j \xi_j + \epsilon_j (p_k + \beta) = \epsilon_j \big(p_k + [\alpha^j + \beta]\big)\,,$$

which is a node of the ϵ_j-scaled lattice. A similar calculation reveals that the map $x \to x_0 - \epsilon_j\xi_j + \rho x$ takes the cell $\frac{\epsilon_j}{\rho}(U+\beta)$ of the $\frac{\epsilon_j}{\rho}$-scaled lattice to the cell $\epsilon_j\big(U + [\alpha^j+\beta]\big)$ of the ϵ_j-scaled lattice, and $\frac{\epsilon_j}{\rho}(\overline{U}_m + \beta) \subset Q_1$ in x-space if and only if $\epsilon_j\big(\overline{U}_m + [\alpha^j+\beta]\big) \subset Q_\rho(x_0) - \epsilon_j\xi_j$ in the image space. It follows from the definition (1.5) of our scaled energy (together with its translation invariance (1.4)) that

$$E^{\epsilon_j/\rho}(w_j^\rho, Q_1) = \rho^{-N} E^{\epsilon_j}(u^{\epsilon_j}, Q_\rho(x_0) - \epsilon_j\xi_j) \tag{3.60}$$

While the blown-up deformation w_j^ρ puts a spotlight on the behavior of u^{ϵ_j} near x_0, its relationship to the affine approximation of u is not obvious. To make that relationship more evident, it is convenient to define

$$w_0(x) = \nabla u(x_0)\cdot x \tag{3.61}$$

and to observe that (3.59) can be rewritten as

$$w_j^\rho(x) = \frac{u^{\epsilon_j}(x_0 - \epsilon_j\xi_j + \rho x) - u(x_0) - \nabla u(x_0)\cdot(\rho x)}{\rho} + w_0(x)\,. \tag{3.62}$$

Notice that the numerator of the first term on the right becomes the affine approximation of u at x_0 if we ignore the small translation $\epsilon_j\xi_j$ and replace u^{ϵ_j} by u.

STEP 3: TAKING THE LIMIT $\epsilon_j \to 0$. In Step 4 we will apply the diagonalization lemma to get a sequence $\rho_j \to 0$ with the following properties:

$$\lim_{j\to\infty} \int_{Q_1} |w_j^{\rho_j}(x) - w_0(x)|^2\,dx = 0\,, \tag{3.63}$$

$$\lim_{j\to\infty} \rho_j^{-N}\mu_j\big(Q_{\rho_j}(x_0) - \epsilon_j\xi_j\big) = \frac{d\mu}{dx}(x_0)\,, \tag{3.64}$$

$$\lim_{j\to\infty} (2\rho_j)^{-N}\mu_j\big(Q_{2\rho_j}(x_0)\big) = \frac{d\mu}{dx}(x_0)\,, \quad \text{and} \tag{3.65}$$

$$\lim_{j\to\infty} \frac{\epsilon_j}{\rho_j} = 0\,. \tag{3.66}$$

The hypothesis of the diagonalization lemma involves a double limit in which ϵ_j tends to 0 first, then ρ tends to 0. Therefore in the present step we lay the groundwork for (3.63) and (3.64) by showing that

(a) if $Q_{2\rho}(x_0) \subset \Omega$ then

$$\lim_{j\to\infty}\int_{Q_1} |w_j^\rho(x) - w_0(x)|^2\,dx = \frac{1}{\rho^{N+2}}\int_{Q_\rho(x_0)} |u(x) - u(x_0) - \nabla u(x_0)\cdot(x - x_0)|^2\,dx\,; \tag{3.67}$$

(b) if in addition $\mu\big(\partial Q_\rho(x_0)\big) = 0$ then

$$\lim_{j\to\infty} \mu_j\big(Q_\rho(x_0) - \epsilon_j\xi_j\big) = \mu\big(Q_\rho(x_0)\big)\,. \tag{3.68}$$

For (3.67), we start by changing variables in (3.62) to get

$$\int_{Q_1} |w_j^\rho(x) - w_0(x)|^2\,dx = \frac{1}{\rho^{N+2}} \int_{Q_\rho(x_0)} |u^{\epsilon_j}(x - \epsilon_j\xi_j) - u(x_0) - \nabla u(x_0)\cdot(x - x_0)|^2\,dx\,.$$

Our task is thus to show that $u^{\epsilon_j}(x - \epsilon_j\xi_j) - u(x)$ converges to 0 in $L^2(Q_\rho(x_0))$. By the triangle inequality

$$|u^{\epsilon_j}(x - \epsilon_j\xi_j) - u(x)| \le |u^{\epsilon_j}(x - \epsilon_j\xi_j) - u(x - \epsilon_j\xi_j)| + |u(x - \epsilon_j\xi_j) - u(x)|\,. \tag{3.69}$$

The first term on the right tends to 0 in $L^2(Q_\rho(x_0))$ since $Q_\rho(x_0) - \epsilon_j\xi_j \subset Q_{2\rho}(x_0) \subset \Omega$ when ϵ_j is sufficiently small, and u^{ϵ_j} tends weakly to u in $H^1(\Omega)$ (which implies strong convergence in $L^2(\Omega)$). To deal with the second term on the right side of (3.69) we use the fact that

$$\int_{Q_\rho(x_0)} |u(x-a) - u(x)|^2\,dx \le Ca^2 \int_{Q_{2\rho}(x_0)} |\nabla u|^2\,dx$$

when a is sufficiently small. Applying this with $a = \epsilon_j\xi_j$, we see that the second term also tends to 0 in $L^2(Q_\rho(x_0))$. This completes the proof of (3.67).

For (3.68) we observe that

$$\big|\mu_j\big(Q_\rho(x_0) - \epsilon_j\xi_j\big)\big| \le \big|\mu_j\big(Q_\rho(x_0) - \epsilon_j\xi_j\big) - \mu_j\big(Q_\rho(x_0)\big)\big| + \big|\mu_j\big(Q_\rho(x_0)\big) - \mu\big(Q_\rho(x_0)\big)\big|\,. \tag{3.70}$$

The second term on the right tends to 0 because the measures μ_j converge weakly to μ and we have assumed that $\mu(\partial Q_\rho(x_0)) = 0$. Indeed, weak convergence implies that $\liminf_{j\to\infty}\mu_j(O) \ge \mu(O)$ when O is open and $\limsup_{j\to\infty}\mu_j(C) \le \mu(C)$ when C is closed, so

$$\mu\big(Q_\rho(x_0)\big) \le \liminf_{j\to\infty} \mu_j\big(Q_\rho(x_0)\big) \le \limsup_{j\to\infty} \mu_j\big(Q_\rho(x_0)\big) \le \limsup_{j\to\infty} \mu_j\big(\overline{Q}_\rho(x_0)\big) \le \mu\big(\overline{Q}_\rho(x_0)\big)\,. \tag{3.71}$$

When $\mu(\partial Q_\rho(x_0)) = 0$ the far left and far right expressions are equal, so each inequality is actually an equality. To deal with the first term on the right side of (3.70) we observe that for any pair of sets A and B,

$$\big|\mu_j(A) - \mu_j(B)\big| \le \mu_j(A \triangle B)$$

where $A\triangle B = (A\setminus B)\cup(B\setminus A)$ is the symmetric difference of A and B. Applying this with $A = Q_\rho(x_0) - \epsilon_j\xi_j$ and $B = Q_\rho(x_0)$, we conclude that for any $\lambda > 0$

$$\big|\mu_j(Q_\rho(x_0) - \epsilon_j\xi_j) - \mu_j(Q_\rho(x_0))\big| \le \mu_j\big(Q_{\rho+\lambda}(x_0)\setminus Q_{\rho-\lambda}(x_0)\big)$$

when ϵ_j is sufficiently small. Since μ_j converges weakly to μ we conclude that

$$\limsup_{j\to\infty} \left|\mu_j(Q_\rho(x_0) - \epsilon_j\xi_j) - \mu_j(Q_\rho(x_0))\right| \le \mu\big(\overline{Q}_{\rho+\lambda}(x_0) \setminus Q_{\rho-\lambda}(x_0)\big) .$$

Now taking the limit $\lambda \to 0$ and using that $\mu(\partial Q_\rho(x_0)) = 0$ we see that the first term on the right in (3.70) tends to 0. This completes the proof of (3.68).

STEP 4: APPLYING THE DIAGONALIZATION LEMMA. We need to avoid the (at most countably many) values of ρ where either $\partial Q_\rho(x_0)$ or $\partial Q_{2\rho}(x_0)$ has nonzero measure under μ. It is therefore convenient to use the discrete version of our diagonalization lemma (see remark 3.3), using a sequence ρ_k converging monotonically to 0 such that

$$\mu\big(\partial Q_{\rho_k}(x_0)\big) = 0 \quad \text{and} \quad \mu\big(\partial Q_{2\rho_k}(x_0)\big) = 0 \quad \text{for all } k . \tag{3.72}$$

We start by observing that

$$\begin{aligned}
&\lim_{k\to 0}\lim_{j\to\infty} \int_{Q_1} |w_j^{\rho_k}(x) - w_0(x)|^2\, dx = 0 ,\\
&\lim_{k\to 0}\lim_{j\to\infty} \rho_k^{-N}\mu_j\big(Q_{\rho_k}(x_0) - \epsilon_j\xi_j\big) = \frac{d\mu}{dx}(x_0) ,\\
&\lim_{k\to 0}\lim_{j\to\infty} (2\rho_k)^{-N}\mu_j\big(Q_{2\rho_k}(x_0)\big) = \frac{d\mu}{dx}(x_0) , \quad \text{and}\\
&\lim_{k\to 0}\lim_{j\to\infty} \frac{\epsilon_j}{\rho_k} = 0 .
\end{aligned}$$

Indeed, the first line is immediate from (3.57) and (3.67); the second is immediate from (3.56) and (3.68); the justification of the third is similar to (but easier than) that of the second; and the last line is obvious. The diagonalization lemma is thus applicable with

$$f(\rho_k, \epsilon_j) = \int_{Q_1} |w_j^{\rho_k}(x) - w_0(x)|^2\, dx + \left|\rho_k^{-N}\mu_j\big(Q_{\rho_k}(x_0) - \epsilon_j\xi_j\big) - \frac{d\mu}{dx}(x_0)\right| + \\ \left|(2\rho_k)^{-N}\mu_j\big(Q_{2\rho_k}(x_0)\big) - \frac{d\mu}{dx}(x_0)\right| + \frac{\epsilon_j}{\rho_k} .$$

It supplies a correspondence $j \mapsto k(j)$ such that (3.63)–(3.66) hold when ρ_j is replaced by $\rho_{k(j)}$. To simplify the notation, *we shall henceforth denote* $\rho_{k(j)}$ *by* ρ_j. (This will lead to no confusion, since we shall make no further use of the original sequence $\{\rho_k\}$ introduced in (3.72).)

We claim that $w_j^{\rho_j}$ converges weakly in $H^1(Q_1)$ to w_0. Since we already know L^2 convergence from (3.63), it suffices to show that $\int_{Q_1} |\nabla w_j^{\rho_j}|^2\, dx$ remains uniformly bounded as $j \to \infty$. To this end we observe that

$$\int_{Q_1} |\nabla w_j^{\rho_j}|^2\, dx = \rho_j^{-N} \int_{Q_{\rho_j}(x_0)-\epsilon_j\xi_j} |\nabla u^{\epsilon_j}|^2\, dx .$$

It is by now familiar that this can be bounded using the key property of our energy that

$$C_2 \int_{\epsilon_j U+\alpha} |\nabla u^{\epsilon_j}|^2\, dx \le E^{\epsilon_j}(u^{\epsilon_j}, \epsilon_j U+\alpha) + D_2|\epsilon_j U+\alpha|\,.$$

Indeed, adding this estimate over all cells $\epsilon_j U+\alpha$ of the ϵ_j-scaled lattice that meet $Q_{\rho_j}(x_0) - \epsilon_j\xi_j$ and using that $\epsilon_j/\rho_j \to 0$, we obtain an estimate of the form

$$\int_{Q_{\rho_j}(x_0)-\epsilon_j\xi_j} |\nabla u^{\epsilon_j}|^2\, dx \le C[E^{\epsilon_j}(u_j^{\epsilon}, Q_{2\rho_j}(x_0)) + \rho_j^N]$$

with a constant C that's independent of j. Finally, we note that

$$\rho_j^{-N} E^{\epsilon_j}(u_j^{\epsilon}, Q_{2\rho_j}(x_0)) \le \rho_j^{-N}\mu_j\big(Q_{2\rho_j}(x_0)\big)\,,$$

which remains bounded as $j \to 0$ by (3.65). These estimates combine to give the desired uniform upper bound on $\int_{Q_1} |\nabla w_j^{\rho_j}|^2\, dx$.

STEP 5: PUTTING IT ALL TOGETHER. In Step 1 we reduced our task to showing that $d\mu/dx \ge \overline{W}(\nabla u)$ almost everywhere. By combining the preceding results, we now show that it holds at x_0. Since $w_j^{\rho_j}$ is defined on the ϵ_j/ρ_j lattice, $\epsilon_j/\rho_j \to 0$, and $w_j^{\rho_j}$ converges weakly to $w_0(x) = \nabla u(x_0)\cdot x$ in $H^1(Q_1)$, we know from lemma 3.1 that

$$\liminf_{j\to 0} E^{\epsilon_j/\rho_j}(w_j^{\rho_j}, Q_1) \ge \overline{W}(\nabla u(x_0))\,.$$

By (3.60) this can be rewritten as

$$\liminf_{j\to 0} \rho_j^{-N} E^{\epsilon_j}(u^{\epsilon_j}, Q_{\rho_j}(x_0) - \epsilon_j\xi_j) \ge \overline{W}(\nabla u(x_0))\,.$$

Now we evaluate the $\liminf$ using (3.64) to obtain the desired conclusion

$$\frac{d\mu}{dx}(x_0) \ge \overline{W}(\nabla u(x_0))\,.$$

□

3.4 The proof of Theorem 2.13

Theorem 2.13 asserts that when a Dirichlet boundary condition is imposed, the Γ-limit is again given by the same effective energy $\int_\Omega \overline{W}(\nabla u)\, dx$. This section provides the proof.

Proof of theorem 2.13. The statement of the theorem requires that the boundary condition $\psi : \partial\Omega \to \mathbb{R}^N$ be Lipschitz continuous. But by Kirzbraun's theorem, such ψ can be extended to a Lipschitz function defined on on all $\mathbb{R}^N$. Therefore we may (and we will) consider that ψ is defined everywhere rather than just on $\partial\Omega$. (Actually, our argument only uses it on a neighborhood of Ω.)

Let us start with the lower bound. It asserts that if $u^\epsilon - \psi \in \mathcal{A}_\epsilon^0(\Omega)$ and $u^\epsilon \rightharpoonup u$ in $H^1(\Omega)$ then

$$\liminf_{\epsilon \to 0} E^\epsilon(u^\epsilon, \Omega) \geq \int_\Omega \overline{W}(\nabla u)\, dx \quad \text{and} \quad u = \psi \text{ at } \partial\Omega\,.$$

The first assertion follows from theorem 2.11, so we only need to prove the second one. Let ψ^ϵ be the piecewise linearization of ψ. (More precisely, ψ^ϵ is the piecewise linearization of the deformation which takes the value $\psi(x^\epsilon)$ at each node x^ϵ of the ϵ-scale lattice.) We know from lemma 2.8 that $|\nabla\psi^\epsilon|$ is uniformly bounded (independent of ϵ) and $|\psi^\epsilon - \psi| \leq C\epsilon$, so it is immediately clear that $\psi^\epsilon \rightharpoonup \psi$ in $H^1(\Omega)$. Since $u^\epsilon - \psi^\epsilon \in \mathcal{A}_\epsilon^0(\Omega)$, this piecewise linear function vanishes at $\partial\Omega$, i.e. it is in $H_0^1(\Omega)$. Since $u^\epsilon - \psi^\epsilon$ converges weakly to $u - \psi$ in $H^1(\Omega)$ and $H_0^1(\Omega)$ is closed under weak H^1 convergence, we conclude that $u = \psi$ at $\partial\Omega$, as desired.

We turn now to finding a recovery sequence. Given any $u \in H^1(\Omega)$ with $u = \psi$ at $\partial\Omega$, we must show the existence of a sequence u^ϵ such that

$$u^\epsilon - \psi \in \mathcal{A}_\epsilon^0 \quad \text{and} \quad E^\epsilon(u^\epsilon, \Omega) \to \int_\Omega \overline{W}(\nabla u)\, dx\,.$$

The sequence provided by the proof of theorem 2.11 is not sufficient, since it doesn't satisfy the first condition. We shall proceed in two steps. In the first, which assumes that $u = \psi$ near $\partial\Omega$, we shall modify the recovery sequence from theorem 2.11 using the method of de Giorgi. In the second step we handle the general case using a density argument. (These arguments are parallel to ones used in [1] for a similar purpose.)

STEP 1: Suppose $u = \psi$ in a neighborhood of $\partial\Omega$, and let $u^\epsilon \rightharpoonup u$ satisfy $E^\epsilon(u^\epsilon, \Omega) \to \int_\Omega \overline{W}(\nabla u)\, dx$. (We showed the existence of such u^ϵ when we proved theorem 2.11.) We now sketch how the method of appendix B lets us modify u^ϵ to obtain a new sequence $\tilde{u}^\epsilon$ with the desired properties.

A key point is that since $u = \psi$ near $\partial\Omega$, we can (and do) choose the set Ω_0' in (B.1) so that $u = \psi$ in $\Omega \setminus \Omega_0'$. Since the desired boundary condition is now ψ rather than λx, we consider the deformation defined at each node of the ϵ-scale lattice by

$$w_i^\epsilon(x) = \psi(x) + \varphi_i(x)(u^\epsilon(x) - \psi(x)) = \varphi_i(x)u^\epsilon(x) + \big(1 - \varphi_i(x)\big)\psi(x) \tag{3.73}$$

rather than the one defined by (B.2). As usual, the piecewise linearization of this deformation will also be called w_i^ϵ.

The arguments that led us to (B.9) extend easily to this setting. Minor adjustments are needed since in appendix B the function λx was its own piecewise linearization, while in the present context $\psi^\epsilon \neq \psi$. However, lemma 2.8 shows that $|\nabla\psi^\epsilon|$ is uniformly bounded, and this is what the argument needs. Consolidating constants, the analogue of (B.9) is

$$E^\epsilon(w_{i(\epsilon)}^\epsilon, \Omega) \leq E^\epsilon(u^\epsilon, \Omega) + C\delta + \frac{C'}{\nu}\left(|\nabla u^\epsilon - \nabla\psi^\epsilon|_{L^2(\Omega\setminus\Omega_0')} + \frac{4\nu^2}{R^2}|u^\epsilon - \psi^\epsilon|_{L^2(\Omega\setminus\Omega_0')}\right)$$

where C and C' do not depend on ϵ, δ, or ν. Since u^ϵ remains bounded in $H^1(\Omega)$, we may (and do) choose $\nu = \nu(\delta)$ so that

$$\frac{C'}{\nu}|\nabla u^\epsilon - \nabla \psi^\epsilon|_{L^2(\Omega\setminus\Omega_0')} \leq \delta$$

for all ϵ. The previous estimate then simplifies to

$$E^\epsilon(w^\epsilon_{i(\epsilon)}, \Omega) \leq E^\epsilon(u^\epsilon, \Omega) + (C+1)\delta + \frac{4C'\nu^2}{R^2}|u^\epsilon - \psi^\epsilon|_{L^2(\Omega\setminus\Omega_0')} \,. \tag{3.74}$$

We claim that $w^\epsilon_{i(\epsilon)}$ converges weakly to u in $H^1(\Omega)$. To show this, we first observe that $E^\epsilon(w^\epsilon_{i(\epsilon)}, \Omega)$ stays uniformly bounded, by (3.74). So the lower bound on our discrete energy can be used to control the L^2 norm of $\nabla w^\epsilon_{i(\epsilon)}$ by arguing as we did for (3.47). (Since $w^\epsilon_{i(\epsilon)} = \psi^\epsilon$ near $\partial\Omega$, we can consider its extension by ψ^ϵ and work in a domain slightly larger than Ω; thus the issue that troubled us in (3.45) is not present here.) Therefore to show weak convergence to u, it suffices to show that

$$\lim_{\epsilon\to 0} |w^\epsilon_{i(\epsilon)} - u|_{L^2(\Omega)} = 0\,. \tag{3.75}$$

This is relatively easy. Notice that in Ω_0' we have $w^\epsilon_{i(\epsilon)} = u^\epsilon$, while in $\Omega \setminus \Omega_0'$, we have $u = \psi$ and $w^\epsilon_{i(\epsilon)}$ is the piecewise linearization of $\psi + \varphi_{i(\epsilon)}(u^\epsilon - \psi)$. Let us write h^ϵ for the piecewise linearization of $\varphi_{i(\epsilon)}(u^\epsilon - \psi)$, or (equivalently) the piecewise linearization of $\varphi_{i(\epsilon)}(u^\epsilon - \psi^\epsilon)$. Then

$$\int_\Omega |w^\epsilon_{i(\epsilon)} - u|^2\, dx = \int_{\Omega\setminus\Omega_0'} |u^\epsilon - u|^2\, dx + \int_{\Omega_0'} |\psi^\epsilon - \psi + h^\epsilon|^2\, dx\,.$$

The first term on the right tends to zero since $u^\epsilon \to u$ in $L^2(\Omega)$, and the second term tends to zero by combining Lemmas 2.8 and 2.9 with the L^2 convergence of u^ϵ to u and the triangle inequality.

Next, we claim that

$$\limsup_{\delta\to 0}\limsup_{\epsilon\to 0} E^\epsilon(w^\epsilon_{i(\epsilon)}, \Omega) = \int_\Omega \overline{W}(\nabla u)\, dx\,. \tag{3.76}$$

This follows easily from (3.74), since $|\psi^\epsilon - \psi| \leq C\epsilon$ by lemma 2.8, while $u^\epsilon \to u$ in L^2 and $u = \psi$ in $\Omega \setminus \Omega_0'$.

We now apply the diagonalization lemma 3.2 with

$$f(\epsilon, \delta) = E^\epsilon(w^\epsilon_{i(\epsilon)}, \Omega) - \int_\Omega \overline{W}(\nabla u)\, dx\,.$$

To be clear about the respective roles of ϵ and δ, we recall that in the definition (3.73) of w^ϵ_i only φ_i depends on δ. So the dependence of $w^\epsilon_{i(\epsilon)}$ on δ is that

$$w^\epsilon_{i(\epsilon)} = \varphi^\delta_{i(\epsilon,\delta)} u^\epsilon + (1 - \varphi^\delta_{i(\epsilon,\delta)})\psi \quad \text{at lattice nodes.}$$

The sequence v^ϵ provided by the diagonalization lemma is obtained by simply taking δ to be a suitable function of ϵ. It is clear that

$$\lim_{\epsilon\to 0} |v^\epsilon - u|_{L^2(\Omega)} = 0$$

since our proof of (3.75) works uniformly in δ. The diagonalization lemma assures us that

$$\limsup_{\epsilon\to 0} E^\epsilon(v^\epsilon,\Omega) \le \int_\Omega \overline{W}(\nabla u)\,dx\,. \tag{3.77}$$

It follows that $\int_\Omega |\nabla v^\epsilon|^2\,dx$ remains bounded, by arguing as we did for $w^\epsilon_{i(\epsilon)}$ a little earlier. Thus v^ϵ converges weakly to u in $H^1(\Omega)$. Now combining (3.77) with lower bound part of theorem 2.11 gives

$$\limsup_{\epsilon\to 0} E^\epsilon(v^\epsilon,\Omega) \le \int_\Omega \overline{W}(\nabla u)\,dx \le \liminf_{\epsilon\to 0} E^\epsilon(v^\epsilon,\Omega)\,.$$

Since $\liminf \le \limsup$, we conclude that $\lim_{\epsilon\to 0} E^\epsilon(v^\epsilon,\Omega) = \int_\Omega \overline{W}(\nabla u)\,dx$. Thus we have achieved the goals of Step 1.

STEP 2: Now consider any $u \in H^1(\Omega)$ satisfying $u = \psi$ at $\partial\Omega$. Since $u - \psi \in H^1_0(\Omega)$ and compactly supported functions are dense in $H^1_0(\Omega)$, for $k = 1, 2, \dots$ we can choose $u_k \in H^1(\Omega)$ such that

$$|u_k - u|_{H^1(\Omega)} \le 2^{-k} \quad \text{and} \quad u_k = \psi \text{ in a neighborhood of } \partial\Omega\,.$$

By Step 1 there is a sequence u^ϵ_k converging weakly to u_k in $H^1(\Omega)$ such that $u^\epsilon_k - \psi \in \mathcal{A}^0_\epsilon(\Omega)$ and

$$\lim_{\epsilon\to 0} E^\epsilon(u^\epsilon_k,\Omega) = \int_\Omega \overline{W}(\nabla u_k)\,dx\,.$$

Using the quadratic growth and Lipschitz properties of $\overline{W}$ we have

$$\lim_{k\to\infty} \int_\Omega \overline{W}(\nabla u_k)\,dx = \int_\Omega \overline{W}(\nabla u)\,dx\,.$$

Therefore we can apply the discrete version of the diagonalization lemma 3.2 with

$$f(\epsilon, 1/k) = \left|E^\epsilon(u^\epsilon_k,\Omega) - \int_\Omega \overline{W}(\nabla u)\,dx\right| + \int_\Omega |u^\epsilon_k - u|^2\,dx$$

to get a sequence $u^\epsilon_{k(\epsilon)}$ that satisfies our Dirichlet boundary condition and converges to u in $L^2(\Omega)$, with

$$\lim_{\epsilon\to 0} E^\epsilon(u^\epsilon_{k(\epsilon)},\Omega) = \int_\Omega \overline{W}(\nabla u)\,dx\,.$$

Moreover, since the discrete energy of $u^\epsilon_{k(\epsilon)}$ stays bounded as $\epsilon \to 0$, we get a uniform H^1 bound on this sequence by arguing as in Step 1. Thus $u^\epsilon_{k(\epsilon)}$ converges weakly to u and its energy converges to the associated effective energy, fullfilling the obligations of a recovery sequence. □

4 Applications to 2D lattice systems of springs

Our framework is applicable to a broad variety of periodic lattice systems. To provide guidance about its use, this section discusses its application to four specific two-dimensional examples. The key point is always to choose a unit cell U and an appropriate energy $E(u, U)$ whose scaled version E^ϵ satisfies our basic conditions (2.12)-(2.15). We must also identify, for each example, the mesh to be used for our piecewise linearization scheme.

As we explained in section 1.2, to avoid unintended degeneracy the energy should include a term penalizing change of orientation. We implement this additively: throughout this section

$$E(u,U) = E_{\text{spr}}(u,U) + E_{\text{pen}}(u,U)\,, \tag{4.1}$$

where E_{spr} is a sum of spring energies and E_{pen} is a change-of-orientation penalty. As discussed in section 1.2, it is natural for $E_{\text{pen}}(u,U)$ to have the form

$$E_{\text{pen}}(u,U) = \sum_{T\in\mathscr{T}} f^{\eta}\big(\det(\nabla u|_T)\big)|T| \tag{4.2}$$

where $\mathscr{T}$ is an appropriately chosen collection of triangles from the mesh used for piecewise linearization and f^{η} is the piecewise constant function defined by (1.7). However other forms are permitted. For our framework to apply, it is sufficient (though not necessary) that E_{pen} be translation-invariant, nonnegative, and bounded above:

$$\begin{aligned} &E_{\text{pen}}(u+c,U) = E_{\text{pen}}(u,U) \text{ when } c \text{ is constant, and} \\ &0 \le E_{\text{pen}}(u,U) \le M \text{ for some finite } M \text{ (independent of } u). \end{aligned} \tag{4.3}$$

Concerning the spring energy E_{spr}: all our examples involve Hookean springs joining selected pairs of lattice nodes. We gave two examples in section 2.1, for the Kagome lattice (2.2) and for a square lattice with long-range interactions (2.3); our other examples will be similar. Since the energy of a Hookean spring is automatically translation-invariant and nonnegative, the only nontrivial requirements on E_{spr} are that it satisfy our upper and lower bounds:

$$E_{\text{spr}}(u,U) \le C_1\Big(|\nabla u|^2_{L^2(U_n)} + |U_n|\Big) \text{ and} \tag{4.4}$$

$$E_{\text{spr}}(u,U) \ge C_2\Big(|\nabla u|^2_{L^2(U)} - D_2|U|\Big)\,, \tag{4.5}$$

where U_n is defined by (2.4) and the constants C_1, C_2, and D_2 must of course be independent of u. Notice that since E_{pen} is assumed to be nonnegative and bounded above, if E_{spr} satisfies these conditions then so does total energy $E = E_{\text{str}} + E_{\text{pen}}$.

Remark 4.1. *A review about the term $|\nabla u|^2_{L^2(U_n)}$ introduced in section 2: while a deformation u takes values only at the nodes of the lattice, we want to treat it as an everywhere-defined piecewise linear function. This is done by triangulating the unit cell U, specifying how u is defined at any "ghost vertices" (see eq. (2.5)), then using the triangulation to define a piecewise linear function. (In our examples, there will actually be no ghost vertices.) The terms $|\nabla u|^2_{L^2(U_n)}$ and $|\nabla u|^2_{L^2(U)}$ on the right hand sides of (4.4) and (4.5) refer to this piecewise linear function.*

For a given lattice system, it is in general a nontrivial task to identify a suitable spring energy $E_{\text{spr}}(u,U)$. If there are springs connecting nodes in the unit cell to nodes in other cells, then E_{spr}

should include the energies of some springs of this type, and the value of n in (4.4) will be bigger than 1. (The square lattice with long-range interactions in fig. 3b has this character.) To satisfy the lower bound (4.5), E_{spr} must include the energies of sufficiently many springs. But it cannot include *too* many, since the total of it and its translates must be the energy of the full lattice system. In some cases, this dichotomy is best handled by letting E_{spr} include just *part* of the energies of some springs. This must, of course, be done with care so that the sum of E_{spr} and its translates counts the energy of each spring exactly once. (We shall proceed this way for the square lattice in section 4.2.)

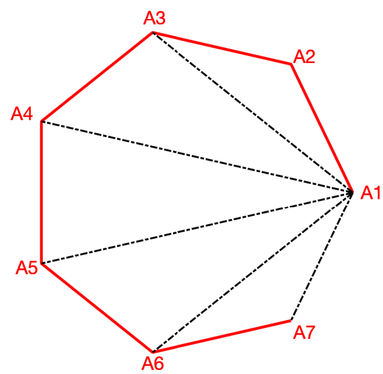


Figure 4: A polygon with $n = 7$: the energy on the red solid edges are counted in $E_{\text{poly}}(u, P_n)$, while the dotted edges are artificial edges to indicate the triangular mesh. Here the vertices are numbered counter-clockwise, however our upper and lower bounds are also valid (with the same proofs) when the vertices are numbered clockwise.

In all our examples, the proofs of the essential inequalities (4.4)–(4.5) rely on a result concerning the spring energy of a convex polygon. We discuss it now in fairly general terms, since this result is also useful for other examples. Consider an n-sided convex polygon P_n as shown in fig. 4, with vertices $A_1, A_2, \ldots, A_n$ where $A_1 \sim A_2$, $A_2 \sim A_3$, $\ldots$, $A_{n-1} \sim A_n$, $A_n \sim A_1$. For a given deformation u that has values at $A_1, A_2, \ldots, A_n$, we consider the following energy

$$E_{\text{poly}}(u, P_n) := \sum_{i=1}^{n-1} \Big(|u(A_{i+1} - u(A_i))| - |A_{i+1} - A_i| \Big)^2 , \tag{4.6}$$

which is the sum of the energies of $n-1$ springs (all except the one connecting A_1 and A_n). We show in Appendix D that this energy has the following upper and lower bounds: for any deformation u that has values at $A_1, \ldots, A_n$,

- there is an upper bound

$$E_{\text{poly}}(u, P_n) \le c_1 \Big(|\nabla u|^2_{L^2(P_n)} + |P_n| \Big) \quad \text{and} \tag{4.7}$$

- there is a lower bound

$$E_{\text{poly}}(u, P_n) \ge c_2 |\nabla u|^2_{L^2(P_n)} - c_3 |P_n| \, , \tag{4.8}$$

with the understanding that $|\nabla u|_{L^2(P_n)}$ is determined by the nodal values of u using the mesh that we are about to discuss. The constants c_1, c_2, c_3 are positive and depend only on the geometry of the polygon. On the right hand side, $|\nabla u|^2_{L^2(P_n)}$ refers to the piecewise triangularization of u using the mesh shown in fig. 4; it consists of $n-2$ triangles: $\Delta A_1 A_2 A_3$, $\Delta A_1 A_3 A_4$, $\ldots$, and $\Delta A_1 A_{n-1} A_n$.

A key feature of estimates (4.7) and (4.8) is that *we only need the spring energy on* $n-1$ *edges* to upper and lower bound the L^2 norm of $|\nabla u|$ on an n-sided polygon. We shall apply these estimates to the four examples considered in this section, and similar arguments work for many other lattice systems of springs.

Remark 4.2. *When using the bounds* (4.7) *and* (4.8)*, it is important to keep in mind that they are not asserted for an arbitrary piecewise linearization scheme; rather, they are asserted only when the right hand side is evaluated using the piecewise linearization scheme specified above.*

4.1 The Kagome and rotating squares metamaterials, viewed as lattices of springs

We start with the Kagome metamaterial and the rotating squares metamaterial as our first illustrative examples, since they have mechanisms but are not entirely degenerate. As we explained in section 1.1, we believe that the soft modes of such systems are best understood as the macroscopic deformations whose effective energy vanishes. It is therefore important to know that there is indeed a well-defined effective energy.

Another interesting feature of these two systems is that besides our spring model, there is also a cut-out model (as we discussed in section 1.1. As a result, it is natural to only penalize change of orientation on *some* of the triangles in our mesh – specifically, those that lie within the material that has been kept in the cut-out model.

4.1.1 The Kagome metamaterial

The Kagome metamaterial was already introduced in section 2.1. Our unit cell and triangular mesh for the Kagome lattice were identified in fig. 3a; for the reader's convenience, they are shown again in fig. 5a. In the cut-out model of this metamaterial the hexagonal regions in fig. 3a are holes, leaving material only in the equilateral triangles. Therefore it is physically natural to penalize change of orientation only in the triangles. Since the unit cell contains two such triangles, ΔAOB and ΔDOC, the set $\mathscr{T}$ in (4.2) should contain just these triangles; thus we propose

$$E_{\text{pen}}(u, U) = f^\eta(\det(\nabla u|_{\Delta AOB})) + f^\eta(\det(\nabla u|_{\Delta DOC})) \tag{4.9}$$

where f^η is given by (1.7) with η sufficiently small.

For $E_{\text{spr}}(u, U)$ we use the energies of the springs associated with the unit cell, which are marked in red in fig. 5a; they lie along AO, BO, CO, DO, DE, and AF. The formula for E_{spr} is thus given by (2.2). To show that these choices fit our framework we must show that E_{spr} satisfies the upper and lower bounds (4.4)–(4.5). This will be done by bounding E_{spr} above or below by sums of energies of polygons then using (4.7) or (4.8).

(a) (b)

Figure 5: The unit cells of the Kagome lattice and the rotating squares lattice: nodes in $\mathcal{V}$ are marked in red; nodes not in $\mathcal{V}$ but used in the energy $E(u,U)$ are marked in cyan; edges whose springs are counted in $E(u,U)$ are marked in red solid lines; artificial edges used to define the triangular mesh are marked in cyan dotted lines; the shaded area is U.

For the upper bound, we observe that all the springs associated with the unit cell U have both ends in $\overline{U}$. Therefore we can take $n = 1$. Since our mesh does not use ghost vertices, we also have $m = 1$. For the upper bound, we observe that the spring energy can be written as the sum of two polygon energies (associated with the triangle ΔBOC and the pentagon P_{FAODE} with vertices F, A, O, D, E), then we apply the upper bound (4.7):

$$\begin{aligned}E_{\text{spr}}(u,U) &= E_{\text{poly}}(u,\Delta BOC) + E_{\text{poly}}(u,P_{FAODE})\\ &\le c_1(\Delta BOC)\Big(|\nabla u|^2_{L^2(\Delta BOC)} + |\Delta BOC|\Big) + c_1(P_{FAODE})\Big(|\nabla u|^2_{L^2(P_{FAODE})} + |P_{FAODE}|\Big)\\ &\le \tilde{c}_1\left(|\nabla u|^2_{L^2(U)} + |U|\right).\end{aligned}$$

For the lower bound we must use a different decomposition, since the one used above does not include triangles ΔAOB and ΔCOD. To cover the missing triangles, we use the decomposition

$$\begin{aligned}E_{\text{spr}}(u,U) = \frac{1}{2}E_{\text{poly}}(u,\Delta AOB) + \frac{1}{2}E_{\text{poly}}(u,\Delta BOC) + \frac{1}{2}E_{\text{poly}}(u,\Delta COD)\\ + \frac{1}{2}E_{\text{poly}}(u,P_{FAODE}) + \frac{1}{2}E_{AF}(u) + \frac{1}{2}E_{DE}(u)\,,\end{aligned}$$

where $E_{AF}(u) = (|u(A) - u(F)| - |A - F|)^2$ is the energy of the spring AF and $E_{DE}(u)$ is the similarly the energy of the spring DE. Using this, we obtain the desired lower bound

$$\begin{aligned}
E_{\text{spr}}(u,U) \geq &\frac{1}{2}\Big(c_2(\Delta AOB)|\nabla u|^2_{L^2(\Delta AOB)} - c_3(\Delta AOB)|\Delta AOB|\Big) \\
&+ \frac{1}{2}\Big(c_2(\Delta BOC)|\nabla u|^2_{L^2(\Delta BOC)} - c_3(\Delta BOC)|\Delta BOC|\Big) \\
&+ \frac{1}{2}\Big(c_2(\Delta COD)|\nabla u|^2_{L^2(\Delta COD)} - c_3(\Delta COD)|\Delta COD|\Big) \\
&+ \frac{1}{2}\Big(c_2(P_{FAODE})|\nabla u|^2_{L^2(P_{FAODE})} - c_3(P_{FAODE})|P_{FAODE}|\Big) \\
\geq &\frac{1}{2}\Big(c_{2,\,\min}|\nabla u|^2_{L^2(U)} - c_{3,\,\max}|U|\Big)
\end{aligned}$$

where $c_{2,\,\min}$ and $c_{3,\,\max}$ are respectively the min and max of the corresponding constants for the polygons under consideration. Since the last expression can easily be rewritten in the desired form $C_2\Big(|\nabla u|^2_{L^2(U)} - D_2|U|\Big)$, we have established the lower bound. Thus the spring model of the Kagome metamaterial fits our framework.

4.1.2 The rotating squares metamaterial

This is perhaps the best-understood mechanism-based mechanical metamaterial, see e.g. [12, 15, 24, 25, 26] for recent developments and many references. Like the Kagome metamaterial, the rotating squares example has both a cut-out model and a spring model. The cut-out model is obtained by patterning a 2D elastic sheet like a checkerboard then removing the white squares. This leaves the black squares meeting one another at corners, which we idealize as hinges where rotation is free. (In a more realistic model the black squares would meet at thin necks, which would permit rotation with very little elastic energy.) This metamaterial has a single mechanism – that is, a one-parameter family of deformations that moves each black square by a rigid motion. Under the mechanism, the originally-square holes become parallelograms (see e.g. Figure 1 in [12]).

The spring lattice version of this structure is obtained by starting with a square lattice then adding diagonals in the "black squares" (to give them rigidity) but not in the "white squares" (which play the role of the holes). The result is shown in Figure 6b, in which the solid edges are all Hookean springs. We like to call this the *rotating squares lattice*. As the unit cell U, it is convenient to use the square with vertices A, C, H, F. As the mesh for piecewise linearizing deformations, we choose the one shown as shown in the figure, which decomposes U into 8 triangles. (Note that the dotted segments BE and DG are not springs; rather, they are merely edges of triangles used for piecewise linearization.)

The natural choice of $E_{\text{pen}}(u,U)$ penalizes change of orientation only in the "black squares" that belong to U. Those squares are P_{BCEO} and P_{DOGF}, so change of orientation should be penalized only on the four triangles $\Delta ABO, \Delta ADO, \Delta OEH$ and ΔOGH.

The spring energy in this example is the aggregate energy of the 10 springs AB, AO, AD, BC, BO,

DO, EO, DF, OG, and OH:

$$E_{\text{spr}}(u,U) = E_{AB}(u) + E_{AO}(u) + E_{BO}(u) + E_{DO}(u) + E_{AD}(u) + E_{BC}(u) + E_{DF}(u) + E_{OG}(u) + E_{OH}(u) + E_{OE}(u)$$

where $E_{AB}(u) = (|u(A) - u(B)| - |A - B|)^2$ is the energy of the spring connecting A and B, etc. It has the feature that when added to the energies of all periodic translates of U we get, as desired, the total energy of the lattice. We note that $n = m = 1$ for this example, since $E_{\text{spr}}(u,U)$ depends only on the values of u in $\overline{U}$ and our piecwise linearization scheme has no ghost vertices.

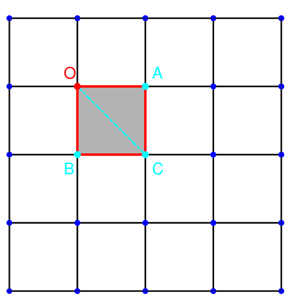
To show that our framework applies to this example, we must show that E_{spr} satisfies our basic upper and lower bounds (4.4) and (4.5). The arguments are similar to those in section 4.1.1. We start by rewriting E_{spr} as

$$E_{\text{spr}}(u,U) = \left[E_{AD}(u) + \frac{1}{2}E_{AO}(u)\right] + \left[E_{AB}(u) + \frac{1}{2}E_{AO}(u)\right] + \left[E_{BC}(u) + E_{BO}(u) + \frac{1}{2}E_{OE}(u)\right] + \left[E_{OD}(u) + E_{DF}(u) + \frac{1}{2}E_{OG}(u)\right] + \frac{1}{2}\left[E_{OG}(u) + E_{OH}(u)\right] + \frac{1}{2}\left[E_{OH}(u) + E_{OE}(u)\right].$$

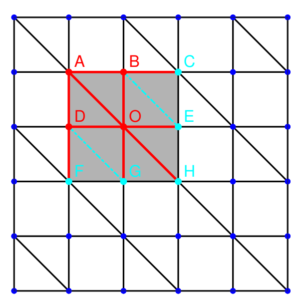
Each term in brackets is bounded above by the energy $E_{\text{poly}}(u, P_n)$ (defined by (4.6)) with P_n being either a triangle or a quadrilateral; similarly, each term in brackets is bounded below by $1/2$ times the energy of a polygon. Using our upper and lower bounds (4.7)–(4.8) and adding, we easily deduce that

$$E_{\text{spr}}(u,U) \leq c_{1,\max}\left(|\nabla u|^2_{L^2(U)} + |U|\right) \quad \text{and} \quad E_{\text{spr}}(u,U) \geq c_{2,\min}|\nabla u|^2_{L^2(U)} - c_{3,\max}|U|$$

where $c_{1,\max}$, $c_{2,\min}$, and $c_{3,\max}$ are the max or min of the corresponding constants for the polygons under consideration. The first inequality has exactly the desired form (4.4) and the second can be rewritten in the desired form (4.5). Thus our framework applies to the rotating squares lattice.



(a)



(b)

Figure 6: Two examples whose nodes are those of a square lattice: (a) the square lattice of springs; (b) the rotating squares lattice. The nodes associated with the unit cell (our set $\mathcal{V}$) are marked in red; nodes not in $\mathcal{V}$ but used in the energy $E(u,U)$ are marked in cyan; springs counted in $E(u,U)$ are marked by red solid lines; artificial edges used only for the triangularization of U are marked by cyan dotted lines; the shaded area is U.

4.2 The square lattice

The square lattice with only nearest-neighbor connections is another interesting example. Unlike the rotating squares lattice depicted in fig. 6b, in this example *none* of the squares have diagonal springs (see fig. 6a). This system has a huge variety of mechanisms. The simplest is a uniform shear, which deforms each square to a parallelogram (all the parallelograms being identical in shape). But there are also many periodic mechanisms, which deform the squares to different-shaped parallelograms. Using suitable periodic mechanisms, one can achieve different compression ratios in the vertical and horizontal directions; moreover, this can be done without any local change of orientation (see for example Figure 1 of [22]). The reader might wonder why it is worthwhile to consider such a degenerate example. The answer lies in the connection between homogenization and soft modes that we discussed in section 1.1. There are systems (including the square lattice and the Kagome lattice) whose mechanisms are not easily enumerated. In such systems, it is not obvious how to define a soft mode. We think a macroscopic deformation u should be considered a soft mode when its effective energy vanishes, i.e. when $\overline{W}(Du)$ vanishes everywhere in Ω. For this proposal to be meaningful, the effective energy must be well-defined even for systems with many mechanisms. The square lattice is a natural example of such a system.

Our choice of the unit cell U for this example is shown in fig. 6a and again in fig. 7a. The mesh used for our piecewise linearization scheme has only two triangles: ΔOAC and ΔOBC. To avoid folding deformations it is natural for E_{pen} to penalize change of orientation on both of these triangles:

$$E_{\text{pen}}(u, U) = f^{\eta}(\det(\nabla u|_{\Delta OAC})) + f^{\eta}(\det(\nabla u|_{\Delta OBC})) \tag{4.10}$$

where f^{η} is defined by (1.7).

The spring energy for this example is

$$E_{\text{spr}}(u, U) = \frac{1}{2}\Big(E_{AO}(u) + E_{BO}(u) + E_{AC}(u) + E_{BC}(u)\Big)\,.$$

The weight $1/2$ assures that when we add the energy of the unit cell and all its translates we count each spring exactly once. (One might ask why we don't take E_{spr} to be the energy of just two springs, for example $E_{OA} + E_{OB}$ or $E_{OB} + E_{BC}$. The answer is that while those choices also get the aggregate spring energy right, neither one satisfies the crucial lower bound (4.5).) Since $E(u, U)$ depends only on nodal values of u in $\overline{U}$ and our piecewise linearization scheme involves no ghost vertices, this example has $n = m = 1$.

We claim that E_{spr} satisfies the required bounds (4.5)-(4.4). To see why, we observe that the spring energy can be rewritten as

$$E_{\text{spr}}(u, U) = \frac{1}{2}E_{\text{poly}}(u, \Delta OAC) + \frac{1}{2}E_{\text{poly}}(u, \Delta OBC)\,, \tag{4.11}$$

in which each term is our polygon energy (4.6) specialized to the indicated triangle:

$$E_{\text{poly}}(u, \Delta OAC) = E_{AO}(u) + E_{AC}(u)\,,$$
$$E_{\text{poly}}(u, \Delta OBC) = E_{BO}(u) + E_{BC}(u)\,.$$

Now the upper and lower bounds (4.7)–(4.8) on the polygon energies combine with (4.11) to give the desired inequalities for E_{spr}, exactly as they did in our discussion of the Kagome and rotating squares lattices. Thus our framework applies to the square lattice.

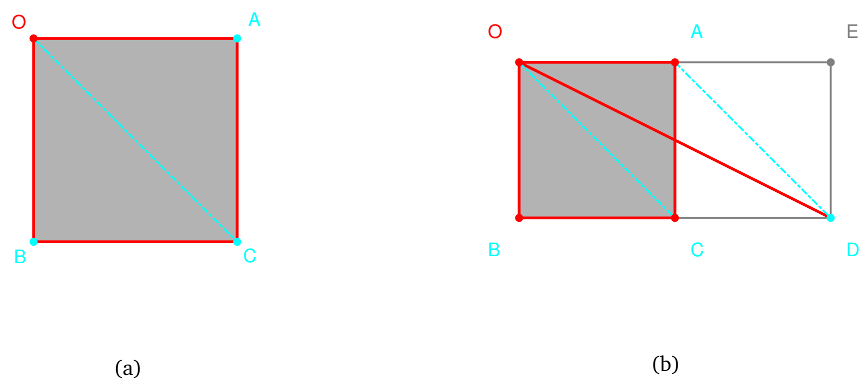


Figure 7: The unit cells of the square lattice and a square lattice with long-range and periodic edges: the nodes associated with the unit cell (that is, the ones in $\mathcal{V}$) are marked in red; nodes not in $\mathcal{V}$ but used in the energy $E(u, U)$ are marked in cyan; edges counted in $E(u, U)$ are marked by red solid lines; artificial edges used only for the triangular mesh are marked by cyan dotted lines; the shaded area is U. The gray edges in (b) are edges of the neighboring unit cell.

4.3 The square lattice with long-range springs

This example is illustrated in fig. 3b. It has all the springs of the square lattice, plus additional springs whose endpoints are *not* in the same translation of $\overline{U}$. Our goal in discussing this example is to show how our framework deals with the presence of long-range springs.

As shown in fig. 7b, we take the unit cell U and the mesh used for piecewise linearization to be exactly as they were for the square lattice. To avoid folding, it is natural to again penalize change of orientation on each triangle, i.e. to again take use (4.10) for E_{pen}.

The spring energy was already identified in section 2.1; we repeat it here for the reader's convenience:

$$E_{\text{spr}}(u, U) = \frac{1}{2}\Big(E_{AO}(u) + E_{BO}(u) + E_{AC}(u) + E_{BC}(u)\Big) + E_{OD}(u)$$

Since $E_{\text{spr}}(u, U)$ depends on $u(D)$ as well as on $u(A)$, $u(B)$, $u(C)$, and $u(D)$, the value of n for this example is clearly greater than 1. Based on the definition (2.4), we see that $n = 2$. Since our piecewise linearization scheme has no ghost vertices, the value of m is also 2.

As usual, we must show that E_{spr} satisfies the upper and lower bounds (4.4) and (4.5). No new work is needed for the lower bound, since the spring energy under discussion here is strictly larger than that of the square lattice, which we have already shown to satisfy the lower bound. (Note that the right side of the lower bound involves only $|\nabla u|_{L^2(U)}$ regardless of the value of n.)

Turning now to the upper bound, we recall that its right hand side involves $|\nabla u|_{L^2(U_2)}$, where U_2 is the union of U and its eight neighbors. We shall actually prove

$$E_{\text{spr}}(u, U) \leq C_1 \Big(|\nabla u|^2_{L^2(U \cup P_{ACD})} + |U \cup P_{ACD}| \Big) \,, \tag{4.12}$$

where P_{ACD} is the triangle with corners A, C, D in fig. 7b. This is stronger than (4.4), since $U \cup P_{ACD}$ is a subset of U_2. We begin with the observation that the energy on OD is bounded by

$$\begin{aligned}(|u(O) - u(D)| - |O - D|)^2 &\leq |u(O) - u(D)|^2 + |O - D|^2 \\ &\leq 2|u(O) - u(A)|^2 + 2|u(A) - u(D)|^2 + |O - D|^2 \,.\end{aligned}$$

Since u is affine on each triangle of our mesh, it is elementary to see that $|u(O)-u(A)|^2 \leq c|\nabla u|^2_{L^2(\Delta OAC)}$ and $|u(A) - u(D)|^2 \leq c'|\nabla u|^2_{L^2(\Delta ADC)}$ where c, c' are suitable constants (independent of u). Since $|O - D|^2$ is also a constant, we easily obtain an inequality of the form

$$E_{OD}(u) \leq C \Big(|\nabla u|^2_{L^2(U \cup P_{ACD})} + |U \cup P_{ACD}| \Big)$$

Combining this with the upper bound proved earlier for the square lattice leads easily to (4.12), completing the proof that this example fits into our framework.

Appendices

A Proofs of Lemma 2.8 and Lemma 2.9

This appendix provides the proofs of lemmas 2.8 and 2.9.

Proof of lemma 2.8. The lemma is stated for any cell $\epsilon U + \alpha$ of the scaled lattice, however to simplify the notation we shall (without loss of generality) take $\alpha = 0$. Our goal is to prove (2.22) – (2.24), which we repeat (with $\alpha = 0$) for the reader's convenience:

$$|u^\epsilon|_{L^\infty(\epsilon U_n)} \leq |\varphi|_{L^\infty(\epsilon U_m)} \,, \quad |\nabla u^\epsilon|_{L^\infty(\epsilon U_n)} \leq C|\nabla \varphi|_{L^\infty(\epsilon U_m)} \,, \quad \text{and} \quad |u^\epsilon - \varphi|_{L^\infty(\epsilon U_n)} \leq C'\epsilon|\nabla \varphi|_{L^\infty(\epsilon U_m)} \,,$$

where u^ϵ refers to the piecewise linearization of the deformation that equals φ at each node of the scaled lattice. We recall that our piecewise linearization scheme uses the scaled version of a triangulation of U that was fixed as part of our framework. The nodes of the lattice must be vertices of the triangulation, but the triangulation can also have other vertices (which we call "ghost vertices" in section 2). If there are ghost vertices, the rules for determining u^ϵ there must also be fixed as part of the framework; moreover they must have the form (2.5)–(2.6). The definition (2.9) of U_m assures that the rules for determining the value of u^ϵ at ghost vertices in ϵU_n use only its values at nodes of the scaled lattice that lie in $\epsilon\overline{U}_m$.

We start by observing that the estimate on $|u^\epsilon - \varphi|$ follows easily from the one on $|\nabla u^\epsilon|$, since $u^\epsilon - \varphi$ vanishes at each node of the scaled lattice. Thus, it suffices to prove the other two estimates.

For simplicity, we shall discuss the 2D case; it will be clear, however, that the same ideas can be used in any space dimension. The estimates are scale-invariant, so it would be sufficient to present the proof only for $\epsilon = 1$; however we shall keep the scale factor ϵ, since setting $\epsilon = 1$ doesn't really simplify matters. Since our triangulation of U_n uses finitely many triangles, it is sufficient to prove

$$|u^\epsilon|_{L^\infty(\epsilon T)} \leq |\varphi|_{L^\infty(\epsilon U_m)} \quad \text{and} \quad |\nabla u^\epsilon|_{L^\infty(\epsilon T)} \leq C_T |\nabla\varphi|_{L^\infty(\epsilon U_m)} \tag{A.1}$$

for each of the triangles T in the triangulation of U_n. (The constant C_T must of course be independent of ϵ.)

The first estimate is easier, so we start with it. As a warm-up, consider the case when all the vertices x_i^ϵ of ϵT are nodes of the scaled lattice. Since ϵT is convex, any point $x^\epsilon \in \epsilon T$ is a convex combination of the vertices: $x^\epsilon = \sum_i \rho_i x_i^\epsilon$ with $0 \leq \rho_i \leq 1$ and $\sum_i \rho_i = 1$. Since u^ϵ is affine on ϵT, we have

$$\begin{aligned} u^\epsilon(x^\epsilon) &= \sum_i \rho_i u^\epsilon(x_i^\epsilon) \\ &= \sum_i \rho_i \varphi(x_i^\epsilon)\,. \end{aligned}$$

Since each vertex belongs to ϵU_m, we conclude the desired estimate $|u^\epsilon(x^\epsilon)| \leq |\varphi|_{L^\infty(\epsilon U_m)}$. For the general case, we must allow some or all of the vertices x_i^ϵ to be ghost nodes. If, say, x_1^ϵ is a ghost node, then (by definition)

$$x_1^\epsilon = \sum_j \theta_j y_j^\epsilon \tag{A.2}$$

where each y_j^ϵ is a node of the scaled lattice in ϵU_m, and the associated evaluation rule is

$$u^\epsilon(x_1^\epsilon) = \sum_j \theta_j u^\epsilon(y_j^\epsilon)\,. \tag{A.3}$$

Since $u^\epsilon = \varphi$ at nodes of the scaled lattice, it follows that

$$|u^\epsilon(x_1)| \leq \sum_j \theta_j |\varphi(y_j^\epsilon)| \leq |\varphi|_{L^\infty(\epsilon U_m)}\,.$$

Thus: we have $|u^\epsilon(x_i^\epsilon)| \leq |\varphi|_{L^\infty(\epsilon U_m)}$ at *each* vertex of ϵT, whether or not it is a ghost vertex. The desired inequality now follows, by the argument we used to warm up.

We turn now to the estimate on $|\nabla u^\epsilon|_{L^\infty(\epsilon T)}$. Let us start once again with the case when all the vertices $\{x_i^\epsilon\}$ of ϵT are nodes of the scaled lattice. Since u^ϵ is affine on ϵT, its gradient on ϵT is characterized by

$$\nabla u^\epsilon = \begin{pmatrix} u^\epsilon(x_1^\epsilon) - u^\epsilon(x_2^\epsilon) & u^\epsilon(x_1^\epsilon) - u^\epsilon(x_3^\epsilon) \end{pmatrix} \begin{pmatrix} x_1^\epsilon - x_2^\epsilon & x_1^\epsilon - x_3^\epsilon \end{pmatrix}^{-1} . \tag{A.4}$$

Since the angles of T are bounded away from 0 there is a bounded, nonsingular matrix M_T such that

$$M_T \begin{pmatrix} x_1 - x_2 & x_1 - x_3 \end{pmatrix} = \begin{pmatrix} |x_1 - x_2| & 0 \\ 0 & |x_1 - x_3| \end{pmatrix} , \tag{A.5}$$

and combining these equations gives

$$\nabla u^\epsilon = \begin{pmatrix} \frac{u^\epsilon(x_1^\epsilon) - u^\epsilon(x_2^\epsilon)}{|x_1^\epsilon - x_2^\epsilon|} & \frac{u^\epsilon(x_1^\epsilon) - u^\epsilon(x_3^\epsilon)}{|x_1^\epsilon - x_3^\epsilon|} \end{pmatrix} M_T . \tag{A.6}$$

Since

$$\frac{|u^\epsilon(x_i^\epsilon) - u^\epsilon(x_j^\epsilon)|}{|x_i^\epsilon - x_j^\epsilon|} = \frac{|\varphi(x_i^\epsilon) - \varphi(x_j^\epsilon)|}{|x_i^\epsilon - x_j^\epsilon|} \leq |\nabla \varphi|_{L^\infty(\epsilon T)} \tag{A.7}$$

we conclude that the (constant) value of ∇u^ϵ on ϵT satisfies

$$|\nabla u^\epsilon| \leq C_T |\nabla \varphi|_{L^\infty(\epsilon T)}$$

with a constant C_T that depends only on the matrix M_T. This is, of course, stronger than (A.1) since the L^∞ norm on the right is restricted to ϵT.

Now suppose two vertices (say, x_1^ϵ and x_2^ϵ) are nodes of the scaled lattice but the third (x_3^ϵ) is a ghost vertex. We recall that $u^\epsilon(x_3^\epsilon)$ is then determined by a specific representation of x_3^ϵ as a convex combination of scaled lattice nodes in $\overline{U}_m$

$$x_3^\epsilon = \sum_j \theta_j y_j^\epsilon , \tag{A.8}$$

and the associated evaluation rule is

$$u^\epsilon(x_3^\epsilon) = \sum_j \theta_j u^\epsilon(y_j^\epsilon) . \tag{A.9}$$

Equations (A.4)–(A.6) are still applicable, and the estimate (A.7) is still available for the difference quotient involving x_1^ϵ and x_2^ϵ, however we must proceed differently for the one that involves x_3^ϵ. We have

$$\begin{aligned} |u^\epsilon(x_1^\epsilon) - u^\epsilon(x_3^\epsilon)| &= \Big| \sum_j \theta_j (\varphi(x_1^\epsilon) - \varphi(y_j^\epsilon)) \Big| \\ &\leq |\nabla \varphi|_{L^\infty(\epsilon U_m)} \sum_j \theta_j |x_1^\epsilon - y_j^\epsilon| \\ &\leq \epsilon d_m |\nabla \varphi|_{L^\infty(\epsilon U_m)} \end{aligned} \tag{A.10}$$

since x_1^ϵ and all the y_j^ϵ's are in $\epsilon\overline{U}_m$. (We have used that since φ is assumed to be Lipschitz continuous, its restriction to $\epsilon\overline{U}_m$ is Lipschitz with constant $|\nabla\varphi|_{L^\infty(\epsilon U_m)}$.) Now let

$$\ell_T^{\min} = \text{minimum length of the sides of } T\,, \tag{A.11}$$

so that

$$|x_1^\epsilon - x_3^\epsilon| \geq \epsilon \ell_T^{\min}\,.$$

Then combining the preceding inequalities gives

$$\frac{|u^\epsilon(x_1^\epsilon) - u^\epsilon(x_3^\epsilon)|}{|x_1^\epsilon - x_3^\epsilon|} \leq \frac{d_m}{\ell_T^{\min}} |\nabla\varphi|_{L^\infty(\epsilon U_m)}\,.$$

Proceeding as before, we obtain an inequality of the form (A.1) with a constant C_T that depends on T only through M_T and $\ell_T^{\min}$.

When the triangle has more than one ghost vertex most of the previous calculation remains intact, however we need an estimate for the difference of u^ϵ between two ghost vertices (as a substitute for (A.10)). Suppose, for example, that x_1^ϵ and x_3^ϵ are both ghost vertices. Keeping our prior notation (A.8)–(A.9) for the rule determining $u^\epsilon(x_3^\epsilon)$, we suppose the corresponding rule determining $u^\epsilon(x_1^\epsilon)$ comes from the expression for x_1^ϵ as a convex combination of scaled lattice nodes in $\epsilon\overline{U}_m$:

$$x_1^\epsilon = \sum_k \theta_k' z_k^\epsilon\,.$$

To take advantage of our previous calculation, we choose a nearby scaled lattice node $x_4^\epsilon \in \epsilon\overline{U}_m$ and use it "as a bridge" between x_1^ϵ and x_3^ϵ:

$$\begin{aligned} u^\epsilon(x_1^\epsilon) - u^\epsilon(x_3^\epsilon) &= \big(u^\epsilon(x_1^\epsilon) - u^\epsilon(x_4^\epsilon)\big) + \big(u^\epsilon(x_4^\epsilon) - u^\epsilon(x_3^\epsilon)\big) \\ &= \sum_k \theta_k' \big(\varphi(z_k^\epsilon) - \varphi(x_4^\epsilon)\big) + \sum_j \theta_j \big(\varphi(x_4^\epsilon) - \varphi(y_j^\epsilon)\big)\,. \end{aligned}$$

Each term on the right has the form we considered in (A.10). Therefore proceeding as before leads to

$$\frac{|u^\epsilon(x_1^\epsilon) - u^\epsilon(x_3^\epsilon)|}{|x_1^\epsilon - x_3^\epsilon|} \leq \frac{2d_m}{\ell_T^{\min}} |\nabla\varphi|_{L^\infty(\epsilon U_m)}\,,$$

from which we once again deduce an inequality of the form (A.1).

The case when x_1^ϵ, x_2^ϵ, and x_3^ϵ are all ghost vertices is essentially the same. (It is natural to use the same "bridge" x_4^ϵ in the estimation of both $u^\epsilon(x_1^\epsilon) - u^\epsilon(x_3^\epsilon)$ and $u^\epsilon(x_1^\epsilon) - u^\epsilon(x_2^\epsilon)$, however our argument does not require this.) □

Proof of lemma 2.9. We once again take $\alpha = 0$ without loss of generality, and we focus again on the 2D setting though it will be clear that the same ideas can be used in any space dimension. Focusing first on (2.25), we shall show that for any triangle T in our triangulation of U_n,

$$|\nabla h^\epsilon|^2_{L^2(\epsilon T)} \le C_T\Big(|u^\epsilon|^2_{L^2(\epsilon U_m)}|\nabla\varphi|^2_{L^\infty(\epsilon U_m)} + |\nabla u^\epsilon|^2_{L^2(\epsilon T)}|\varphi|^2_{L^\infty(\epsilon T)}\Big)\,. \tag{A.12}$$

Here u^ϵ is the piecewise linearization of a deformation defined at all lattice nodes in $\epsilon\overline{U}_m$, φ is a Lipschitz continuous function, and h^ϵ is the piecewise linearization of a deformation that equals $u^\epsilon\varphi$ at each lattice node in $\epsilon\overline{U}_m$. The constant C_T in (A.12) will depend on the shape of T and the details of our piecewise linearization scheme, but it will be independent of ϵ, u^ϵ, and φ. The estimate (2.25) follows immediately by summing (A.12) over the (finitely many) triangles T in the triangulation of U_n.

We start with some preliminary observations. The first is that if the triangle ϵT has vertices $\{x_i^\epsilon\}_{i=1}^3$ and u^ϵ is an affine function on ϵT then ∇u^ϵ (which is constant) satisfies

$$|\nabla u^\epsilon|^2 \sim \sum_{i\neq j}\frac{|u^\epsilon(x_i^\epsilon)-u^\epsilon(x_j^\epsilon)|^2}{|x_i^\epsilon - x_j^\epsilon|^2} \tag{A.13}$$

in the sense that each side is less than or equal to an ϵ-independent constant times the other. This is an immediate consequence of (A.6).

Our second observation is that

$$\frac{1}{|\epsilon T|}\int_{\epsilon T}|u^\epsilon|^2\,dx \sim \sum_i |u^\epsilon(x_i^\epsilon)|^2\,. \tag{A.14}$$

Since this estimate is scale invariant, it suffices to prove it when $\epsilon = 1$. Writing x_i for the vertices and u for the affine deformation, we may represent u using introduce barycentric coordinates. This means writing

$$u(x) = \sum_i u(x_i)\lambda_i(x)$$

where λ_i is the affine function on T with value 1 at x_i and 0 at the other vertices. Evidently

$$\int_T |u(x)|^2\,dx = \sum_{i,j} u(x_i)\cdot u(x_j)\int_T \lambda_i(x)\lambda_j(x)\,dx\,. \tag{A.15}$$

The right side is a quadratic form in $\{u(x_i)\}$. We see from (A.15) that it is positive definite, so

$$\sum_{i,j} u(x_i)\cdot u(x_j)\int_T \lambda_i(x)\lambda_j(x)\,dx \sim \sum_i |u(x_i)|^2\,. \tag{A.16}$$

Combining (A.15) and (A.16), we get the $\epsilon = 1$ version of (A.14).

Turning now to our main task, we begin by considering the case when all three vertices x_i^ϵ of ϵT are nodes of the scaled lattice. Then by (A.13)

$$|\nabla h^\epsilon|^2 \sim \sum_{i\neq j}\frac{|u^\epsilon(x_i^\epsilon)\varphi(x_i^\epsilon) - u^\epsilon(x_j^\epsilon)\varphi(x_j^\epsilon)|^2}{|x_i^\epsilon - x_j^\epsilon|^2}\,.$$

For each pair $i \neq j$ we have

$$
\begin{aligned}
|u^\epsilon(x_i^\epsilon)\varphi(x_i^\epsilon) - u^\epsilon(x_j^\epsilon)\varphi(x_j^\epsilon)| &\leq |u^\epsilon(x_i^\epsilon)|\,|\varphi(x_i^\epsilon) - \varphi(x_j^\epsilon)| + |u^\epsilon(x_i^\epsilon) - u^\epsilon(x_j^\epsilon)|\,|\varphi(x_j^\epsilon)| \\
&\leq |u^\epsilon(x_i^\epsilon)|\,|\nabla\varphi|_{L^\infty(\epsilon T)}|\,|x_i^\epsilon - x_j^\epsilon| + |u^\epsilon(x_i^\epsilon) - u^\epsilon(x_j^\epsilon)|\,|\varphi|_{L^\infty(\epsilon T)}\,.
\end{aligned}
$$

Since u^ϵ is affine on ϵT our first observation applies to it, so

$$
\frac{|u^\epsilon(x_i^\epsilon)\varphi(x_i^\epsilon) - u^\epsilon(x_j^\epsilon)\varphi(x_j^\epsilon)|}{|x_i^\epsilon - x_j^\epsilon|} \leq C_T\Big(|u^\epsilon(x_i^\epsilon)|\,|\nabla\varphi|_{L^\infty(\epsilon T)} + |\nabla u^\epsilon|\,|\varphi|_{L^\infty(\epsilon T)}\Big)\,,
$$

where on the right hand side $|\nabla u^\epsilon|$ is the norm of the constant matrix $\nabla u^\epsilon\big|_T$. Squaring both sides and applying our second observation to u^ϵ, we conclude that

$$
\frac{|u^\epsilon(x_i^\epsilon)\varphi(x_i^\epsilon) - u^\epsilon(x_j^\epsilon)\varphi(x_j^\epsilon)|^2}{|x_i^\epsilon - x_j^\epsilon|^2} \leq C_T\Big(\frac{1}{|\epsilon T|}|u|^2_{L^2(\epsilon T)}|\nabla\varphi|^2_{L^\infty(\epsilon T)} + |\nabla u^\epsilon|^2|\varphi|^2_{L^\infty(\epsilon T)}\Big)\,. \tag{A.17}
$$

(Here and below, we permit the constant C_T to change from line to line, however it always represents an ϵ-independent constant depending only on the triangle T and our piecewise linearization scheme.) We integrate over ϵT and sum over $i \neq j$ to get

$$
|\nabla h^\epsilon|^2_{L^2(\epsilon T)} \leq C_T\Big(|u^\epsilon|^2_{L^2(\epsilon T)}|\nabla\varphi|^2_{L^\infty(\epsilon T)} + |\nabla u^\epsilon|^2_{L^2(\epsilon T)}|\varphi|^2_{L^\infty(\epsilon T)}\Big)\,.
$$

This is stronger than (A.12), since on the right the L^∞ norms are only over ϵT.

It remains to consider the case when some or all the vertices of ϵT are ghost vertices. We still have

$$
|\nabla h^\epsilon|^2 \sim \sum_{i\neq j}\frac{|h^\epsilon(x_i^\epsilon) - h^\epsilon(x_j^\epsilon)|^2}{|x_i^\epsilon - x_j^\epsilon|^2}
$$

but the evaluation of $h^\epsilon(x_i^\epsilon) - h^\epsilon(x_j^\epsilon)$ must proceed differently when one or both of $x_i^\epsilon, x_j^\epsilon$ are ghost nodes. To simplify the notation let us take $i = 1$ and $j = 3$, and to see the idea in its simplest form let us suppose for now that x_3^ϵ is a ghost node but x_1^ϵ is not. Recall that the rule determining $h^\epsilon(x_3)$ is then of the form (A.8)–(A.9), so

$$
h^\epsilon(x_1^\epsilon) - h^\epsilon(x_3^\epsilon) = u^\epsilon(x_1^\epsilon)\varphi(x_1^\epsilon) - \sum_j \theta_j u^\epsilon(y_j^\epsilon)\varphi(y_j^\epsilon) \tag{A.18}
$$

where $\{y_j^\epsilon\}$ are certain nodes of the scaled lattice that lie in $\overline{U}_m$. Since u^ϵ is itself the piecewise linearization of a deformation defined at scaled lattice nodes, we also have

$$
u^\epsilon(x_3^\epsilon) = \sum_j \theta_j u^\epsilon(y_j^\epsilon)\,.
$$

Combining these relations gives

$$h^\epsilon(x_1^\epsilon) - h^\epsilon(x_3^\epsilon) = \left[u^\epsilon(x_1^\epsilon)\varphi(x_1^\epsilon) - u^\epsilon(x_3^\epsilon)\varphi(x_3^\epsilon)\right] + \sum_j \theta_j u^\epsilon(y_j^\epsilon)\big(\varphi(x_3^\epsilon) - \varphi(y_j^\epsilon)\big)\,. \tag{A.19}$$

The term in square brackets is of the form we considered when there were no ghost nodes, so we have already estimated it. As for the other term: we have

$$\Big|\sum_j \theta_j u^\epsilon(y_j^\epsilon)\big(\varphi(x_3^\epsilon) - \varphi(y_j^\epsilon)\big)\Big| \le \epsilon d_m |\nabla\varphi|_{L^\infty(\epsilon U_m)} \max_j |u^\epsilon(y_j^\epsilon)|\,,$$

whence (using our second observation and also (A.11))

$$\frac{1}{|x_1^\epsilon - x_3^\epsilon|^2}\left(\sum_j \theta_j u^\epsilon(x_3^\epsilon)\big(\varphi(x_3^\epsilon) - \varphi(y_j^\epsilon)\big)\right)^2 \le C_T |\nabla\varphi|^2_{L^\infty(\epsilon U_m)} \sum_{\substack{T' \in \text{ triangulation}\\ \text{of } U_m}} \frac{1}{|\epsilon T'|}\int_{\epsilon T'} |u^\epsilon(x)|^2\,dx\,.$$

Combining this with (A.17) and (A.19) gives

$$\frac{|h^\epsilon(x_1^\epsilon) - h^\epsilon(x_3^\epsilon)|^2}{|x_1^\epsilon - x_3^\epsilon|^2} \le C_T\Big(\max_{\substack{T' \in \text{ triangulation}\\ \text{of U}_\text{m}}} \frac{1}{|\epsilon T'|}|u^\epsilon|^2_{L^2(\epsilon T')}|\nabla\varphi|^2_{L^\infty(\epsilon U_m)} + |\nabla u^\epsilon|^2|\varphi|^2_{L^\infty(\epsilon T)}\Big) \tag{A.20}$$

where on the right hand side $|\nabla u^\epsilon|$ is the norm of the constant matrix $\nabla u^\epsilon|_T$.

The same estimate (A.20) is also valid when both x_1 and x_3 are ghost nodes. Indeed, if the rule defining $h^\epsilon(x_1^\epsilon)$ is associated with (A.2) then (A.18) is replaced by

$$h^\epsilon(x_1^\epsilon) - h^\epsilon(x_3^\epsilon) = \sum_k \theta'_k u^\epsilon(z_k^\epsilon)\varphi(z_k^\epsilon) - \sum_j \theta_j u^\epsilon(y_j^\epsilon)\varphi(y_j^\epsilon)\,.$$

Since we also have $u^\epsilon(x_1^\epsilon) = \sum_k \theta'_k u^\epsilon(z_k^\epsilon)$, this can be rewritten as

$$\begin{aligned} h^\epsilon(x_1^\epsilon) - h^\epsilon(x_3^\epsilon) =& \left[u^\epsilon(x_1^\epsilon)\varphi(x_1^\epsilon) - u^\epsilon(x_3^\epsilon)\varphi(x_3^\epsilon)\right] + \\ & \sum_j \theta_j u^\epsilon(y_j^\epsilon)\big(\varphi(x_3^\epsilon) - \varphi(y_j^\epsilon)\big) + \sum_k \theta'_k u^\epsilon(z_k^\epsilon)\big(\varphi(z_k^\epsilon) - \varphi(x_1^\epsilon)\big)\,. \end{aligned}$$

Each term is of a type we have already discussed, so arguing as before we obtain once again an estimate of the form (A.20).

Finally, our assertion (A.12) follows easily from these results. Indeed, we have shown that the right hand side of (A.20) estimates $|h^\epsilon(x_i^\epsilon) - h^\epsilon(x_j^\epsilon)|^2/|x_i^\epsilon - x_j^\epsilon|^2$ for every pair of vertices of ϵT. Integrating this estimate over ϵT and using our first observation leads immediately to (A.12), since

$$\max_{\substack{T' \in \text{ triangulation}\\ \text{of } U_m}} \frac{|\epsilon T|}{|\epsilon T'|}$$

is an ϵ-independent constant that depends only on the triangle T and our piecewise linearization scheme. The proof of (2.25) is now complete.

We turn now to the Lemma's second assertion, (2.26). It clearly suffices to show that for each triangle T in our triangulation of U_n we have

$$|h^\epsilon|^2_{L^2(\epsilon T)} \leq C|u^\epsilon|^2_{L^2(\epsilon U_m)}|\varphi|^2_{L^\infty(\epsilon U_m)}\,. \tag{A.21}$$

By (A.14) we have

$$|h^\epsilon|^2_{L^2(\epsilon T)} \leq C|\epsilon T|\sum_i |h^\epsilon(x_i^\epsilon)|^2 \tag{A.22}$$

where x_i^ϵ are the vertices of ϵT. If all the x_i^ϵ are nodes of the lattice then things are very simple: $|h^\epsilon(x_i^\epsilon)| \leq |u^\epsilon(x_i^\epsilon)||\varphi|_{L^\infty(\epsilon T)}$, so another application of (A.14) gives

$$|h^\epsilon|^2_{L^2(\epsilon T)} \leq C|u^\epsilon|^2_{L^2(\epsilon T)}|\varphi|^2_{L^\infty(\epsilon T)}\,,$$

which is better than (A.21). If, however, some or all the x_i^ϵ are ghost vertices, then we must work a little harder. Suppose, for example, that x_1^ϵ is a ghost vertex, and that the piecewise linearization rule uses (A.2)–(A.3). Then

$$\begin{aligned}
|h^\epsilon(x_1^\epsilon)| &= \big|\sum_j \theta_j \varphi(y_j^\epsilon)u^\epsilon(y_j^\epsilon)\big| \\
&\leq \big(\sum_j \theta_j^2\big)^{1/2}\,|\varphi|_{L^\infty(\epsilon U_m)}\,\big(\sum_j |u^\epsilon(y_j^\epsilon)|^2\big)^{1/2} \\
&\leq |\varphi|_{L^\infty(\epsilon U_m)}\,\big(\sum_j |u^\epsilon(y_j^\epsilon)|^2\big)^{1/2}
\end{aligned}$$

using that $\sum_j \theta_j^2 \leq \sum_j \theta_j = 1$ since $0 \leq \theta_j \leq 1$. Squaring this, using that the vertices of our triangulation include all lattice nodes, and using (A.14) once again, we have

$$|h^\epsilon(x_1^\epsilon)|^2 \leq C|\varphi|^2_{L^\infty(\epsilon U_m)} \sum_{\substack{T' \in \text{ triangulation} \\ \text{of } U_m}} \frac{1}{|\epsilon T'|}\int_{\epsilon T'} |u^\epsilon(x)|^2\,dx\,.$$

Treating each ghost vertex of ϵT this way and using (A.22) we conclude that

$$|h^\epsilon|^2_{L^2(\epsilon T)} \leq C|\varphi|^2_{L^\infty(\epsilon U_m)}\int_{\epsilon U_m}|u^\epsilon|^2\,dx\left(\max_{\substack{T' \in \text{ triangulation} \\ \text{of } U_m}} \frac{|\epsilon T|}{|\epsilon T'|}\right)\,.$$

Since the max that appears on the far right is a constant independent of ϵ, this establishes (A.21). □

B Details of Step 3 in the proof of Lemma 3.1

We recall that Step 2 in the proof of lemma 3.1 part (b) established the lower bound (3.2) when $u^\epsilon - \lambda x \in \mathcal{A}_0^\epsilon(\Omega)$. Our task here is to prove that the same lower bound holds without imposing such an "affine boundary condition," provided that u^ϵ converges weakly to $u(x) = \lambda x$ in $H^1(\Omega)$. As already noted in section 3.1, the proof uses a well-known argument due to De Giorgi, whose main novelty in our setting is its need for lemma 2.9.

We begin with some preliminaries. First, for arbitrarily small δ, we choose two open subsets of Ω that are both compactly supported in Ω with the following two properties:

$$\Omega_0' \subset\subset \Omega_0 \subset\subset \Omega \qquad \text{and} \qquad |\Omega \setminus \Omega_0'| \leq \delta\,. \tag{B.1}$$

Second, we introduce a nested family of sets Ω_i that contain Ω_0, by taking (for any positive integer ν, which will eventually tend to infinity)

$$\Omega_i = \{x \in \Omega \,\big|\, \text{dist}(x, \Omega_0) < \frac{i}{\nu} R\} \qquad i = 1, 2, \ldots, \nu\,, \quad \text{where} \quad R = \frac{1}{2}\text{dist}(\partial\Omega, \Omega_0)\,.$$

We thus obtain a sequence of sets with $\Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_\nu \subset \Omega$, as shown in Figure 8. Next, we choose corresponding cut-off functions $\varphi_i(x)$ such that $0 \leq \varphi \leq 1$ with

$$\varphi_i(x) = \begin{cases} 1 & x \in \Omega_{i-1} \\ 0 & x \in \Omega \setminus \Omega_i \end{cases} \qquad \text{and} \qquad \big|\nabla\varphi_i\big|_{L^\infty(\Omega)} \leq \frac{2\nu}{R}\,.$$

The upper bound on the L^∞ norm of $\nabla\varphi$ is feasible, since for every pair $x \in \Omega_{i-1}$ and $y \in \partial\Omega_i$, their distance is lower bounded by R/ν.

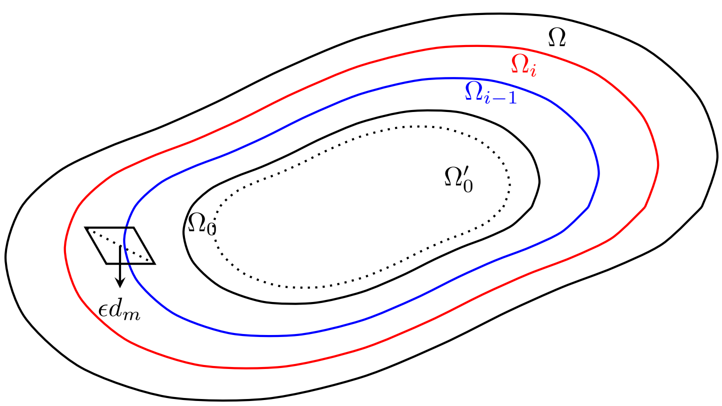


Figure 8: An illustratration of the nested sets $\{\Omega_i\}, i = 1, 2, \ldots, \nu - 3$. The parallelogram (lower left) is a scaled cell $\epsilon U_m + \alpha$ with $\alpha \in R_\epsilon(S_i^\epsilon)$; the dotted diagonal line indicates the largest distance between two points in the $\epsilon U_m + \alpha$, which is ϵd_m.

Finally, we introduce the deformations $w_i^\epsilon(x)$, defined at nodes of our ϵ-scale lattice by

$$w_i^\epsilon(x) = \lambda x + \varphi_i(x)(u^\epsilon - \lambda x) = \varphi_i(x) u^\epsilon + \big(1 - \varphi_i(x)\big)\lambda x\,. \tag{B.2}$$

Notice that each $w_i^\epsilon(x)$ is "affine at $\partial\Omega$" (in the sense that $w_i^\epsilon - \lambda x \in \mathcal{A}_\epsilon^0(\Omega)$), since

$$w_i^\epsilon(x) = \begin{cases} u^\epsilon & x \in \Omega_{i-1} \\ \lambda x & x \in \Omega \setminus \Omega_i \,. \end{cases}$$

By Step 2 of the proof of lemma 3.1 part (b), we know that

$$\liminf_{\epsilon \to 0} E^\epsilon(w_i^\epsilon, \Omega) \geq \left|\Omega\right| \overline{W}(\lambda) \,. \tag{B.3}$$

Our plan is to show that if i is chosen properly, $E^\epsilon(w_i^\epsilon, \Omega)$ is close to $E^\epsilon(u^\epsilon, \Omega)$. To prepare for this argument, we observe that the energy $E^\epsilon(w_i^\epsilon, \Omega)$ has (roughly speaking) three parts: the energy of u^ϵ inside Ω_{i-1}, the energy of λx outside Ω_i, and the energy associated with w_i^ϵ in the layer $\Omega_i \setminus \Omega_{i-1}$. To make this more precise, we introduce the sets

$$S_i^\epsilon = \{x \in \Omega \,\big|\, \text{dist}(x, \overline{\Omega_i} \setminus \Omega_{i-1}) < 2\epsilon d_m\}, \qquad i = 1, 2, \ldots, \nu - 3 \,,$$

where d_m is defined in (2.10). They are useful because

$$\begin{aligned} E^\epsilon(w_i^\epsilon, \Omega) &\leq E^\epsilon(w_i^\epsilon, \Omega_{i-1}) + E^\epsilon(w_i^\epsilon, \Omega \setminus \overline{\Omega}_i) + E^\epsilon(w_i^\epsilon, S_i^\epsilon) \\ &= E^\epsilon(u^\epsilon, \Omega_{i-1}) + E^\epsilon(\lambda x, \Omega \setminus \overline{\Omega}_i) + E^\epsilon(w_i^\epsilon, S_i^\epsilon). \end{aligned} \tag{B.4}$$

This upper bound holds since if a scaled cell $\epsilon U + \alpha$ has the property that $\epsilon U_m + \alpha$ is neither compactly included in Ω_{i-1} nor in $\Omega \setminus \overline{\Omega_i}$, then $\epsilon \overline{U}_m + \alpha$ must intersect $\overline{\Omega_i} \setminus \Omega_{i-1}$. When this happens, $\alpha \in R_\epsilon(S_i^\epsilon)$ (see Figure 8 for a visualization of this). These sets have the following properties for ϵ sufficiently small:

(i) $\cup_{i=1}^{\nu-3} S_i^\epsilon \subset \Omega \setminus \Omega_0'$;

(ii) $R_\epsilon(S_i^\epsilon) \cap R_\epsilon(S_j^\epsilon) \neq \emptyset$ if and only if $|i-j| = 1$;

(iii) $\cup_{i=1}^{\nu-3} R_\epsilon(S_i^\epsilon) \subset R_\epsilon(\Omega \setminus \Omega_0')$;

(iv) $S_i^\epsilon \cap S_j^\epsilon \neq \emptyset$ if and only if $|i-j| = 1$;

(v) $\sum_{i=1}^{\nu-3} |S_i^\epsilon| \leq 2|\Omega \setminus \Omega_0'|$.

We are now ready to show the desired lower bound. The key idea is to show that the right hand side of (B.4) is upper bounded by $E^\epsilon(u^\epsilon, \Omega)$ and some small terms. Combining this with (B.3) will then give the desired lower bound for the energy of u^ϵ. So our task is to estimate the right hand side of (B.4). The first term is easy: we have

$$E^\epsilon(u^\epsilon, \Omega_{i-1}) \leq E^\epsilon(u^\epsilon, \Omega) \,. \tag{B.5}$$

The second term is also easy: by lemma 2.3 we have

$$E^\epsilon(\lambda x, \Omega \setminus \overline{\Omega_i}) \le E^\epsilon(\lambda x, \Omega \setminus \overline{\Omega_0'}) \le C_1(2n-1)^N(|\lambda|^2+1)|\Omega \setminus \overline{\Omega_0'}| \le C_1(2n-1)^N\delta(|\lambda|^2+1)\,. \quad \text{(B.6)}$$

The third term in (B.4) is, however, more difficult; in fact, if i is held fixed then it cannot be adequately controlled. However, we will show the existence of a choice $i(\epsilon)$ for every ϵ such that $E^\epsilon(w^\epsilon_{i(\epsilon)}, S^\epsilon_{i(\epsilon)})$ is adequately controlled. The idea is relatively simple: we first estimate the average $\frac{1}{\nu-3}\sum_{i=1}^{\nu-3} E^\epsilon(w_i^\epsilon, S_i^\epsilon)$, then take $i(\epsilon)$ corresponding to the smallest of the terms that were averaged.

We start by estimating $E^\epsilon(w_i^\epsilon, S_i^\epsilon)$ using our basic upper bound (2.14) on the energy of the unit cell:

$$\begin{aligned} E^\epsilon(w_i^\epsilon, S_i^\epsilon) &= \sum_{\alpha\in R_\epsilon(S_i^\epsilon)} E^\epsilon(w_i^\epsilon, \epsilon U+\alpha) \\ &\le C_1 \sum_{\alpha\in R_\epsilon(S_i^\epsilon)} \Big(\big|\epsilon U_n\big| + \big|\nabla w_i^\epsilon\big|^2_{L^2(\epsilon U_n+\alpha)}\Big)\,. \end{aligned}$$

The right hand side of this bound refers, as usual, to the piecewise linearization of w_i^ϵ. Remembering from (B.2) that $w_i^\epsilon(x) = \lambda x + \varphi_i(x)(u^\epsilon - \lambda x)$ at the nodes of the scaled lattice, and recalling that the linear function λx is its own piecewise linearization, we apply lemma 2.9 to get

$$|\nabla w_i^\epsilon|^2_{L^2(\epsilon U_n+\alpha)} \le C\Big(|\lambda|^2|\epsilon U_m| + |\nabla u^\epsilon - \lambda|^2_{L^2(\epsilon U_m+\alpha)} + \frac{4\nu^2}{R^2}|u^\epsilon-\lambda x|^2_{L^2(\epsilon U_m+\alpha)}\Big)\,,$$

where the constant $C \ge 1$ depends only on our piecewise linearization scheme. Combining the preceding estimates gives

$$\begin{aligned} E^\epsilon(w_i^\epsilon, S_i^\epsilon) &\le CC_1 \sum_{\alpha\in R_\epsilon(S_i^\epsilon)} \Big((1+|\lambda|^2)\big|\epsilon U_m\big| + |\nabla u^\epsilon - \lambda|^2_{L^2(\epsilon U_m+\alpha)} + \frac{4\nu^2}{R^2}|u^\epsilon - \lambda x|^2_{L^2(\epsilon U_m+\alpha)}\Big) \\ &\le CC_1(2m-1)^N\Big((1+|\lambda|^2)\big|S_i^\epsilon\big| + |\nabla u^\epsilon-\lambda|^2_{L^2(S_i^\epsilon)} + \frac{4\nu^2}{R^2}|u^\epsilon-\lambda x|^2_{L^2(S_i^\epsilon)}\Big)\,. \end{aligned}$$

The second line holds since each $|\nabla u^\epsilon|^2_{L^2(\epsilon U+\alpha)}$ with $\alpha \in R_\epsilon(S_i^\epsilon)$ is appears in $|\nabla u^\epsilon|^2_{L^2(\epsilon U_m+\beta)}$ for some β at most $(2m-1)^N$ times. We now use that for sufficiently small ϵ the sets S_i^ϵ and S_j^ϵ are disjoint unless $|i-j| = 1$ (see bullet (iv)), and similarly $R_\epsilon(S_i^\epsilon)$ and $R_\epsilon(S_j^\epsilon)$ are disjoint unless $|i-j|=1$ (see bullet (ii)). This leads to the following estimate on the average of $E^\epsilon(w_i^\epsilon, S_i^\epsilon)$ from $i=1$ to $\nu-3$:

$$\begin{aligned} \tfrac{1}{\nu-3}\textstyle\sum_{i=1}^{\nu-3} E^\epsilon(w_i^\epsilon, S_i^\epsilon) &\le \tfrac{CC_1(2m-1)^N}{\nu-3}\sum_{i=1}^{\nu-3}\Big((1+|\lambda|^2)|S_i^\epsilon| + |\nabla u^\epsilon-\lambda|^2_{L^2(S_i^\epsilon)} + \tfrac{4\nu^2}{R^2}|u^\epsilon-\lambda x|^2_{L^2(S_i^\epsilon)}\Big) \\ &\le \tfrac{2CC_1(2m-1)^N}{\nu-3}\Big((1+|\lambda|^2)\big|\Omega\setminus\Omega_0'\big| + |\nabla u^\epsilon - \lambda|^2_{L^2(\Omega\setminus\Omega_0')} + \tfrac{4\nu^2}{R^2}|u^\epsilon-\lambda x|^2_{L^2(\Omega\setminus\Omega_0')}\Big)\,. \quad \text{(B.7)} \end{aligned}$$

There is a factor of 2 in (B.7) because we at most cover the whole area $\Omega\setminus\Omega_0'$ twice when summing over all $i = 1, 2, \ldots, \nu-3$ (see bullet (v)). Finally, we choose

$$i(\epsilon) = \arg\min E^\epsilon(w_i^\epsilon, S_i^\epsilon)$$

so that

$$E^\epsilon(w^\epsilon_{i(\epsilon)}, S^\epsilon_i) \le \frac{1}{\nu-3}\sum_{i=1}^{\nu-3} E^\epsilon(w^\epsilon_i, S^\epsilon_i)\,. \tag{B.8}$$

Combining (B.4)–(B.8), we obtain

$$\begin{aligned} E^\epsilon(w^\epsilon_{i(\epsilon)}, \Omega) \le\ & E^\epsilon(u^\epsilon, \Omega) + C_1\delta(|\lambda|^2+1) \\ &+ \frac{2CC_1(2m-1)^N}{\nu-3}\Big((1+|\lambda|^2)\big|\Omega\setminus\Omega_0'\big| + |\nabla u^\epsilon - \lambda|^2_{L^2(\Omega\setminus\Omega_0')} + \frac{4\nu^2}{R^2}|u^\epsilon - \lambda x|^2_{L^2(\Omega\setminus\Omega_0')}\Big)\,. \end{aligned} \tag{B.9}$$

We know from Step 2 of the proof of lemma 3.1 part (b) that

$$|\Omega|\overline{W}(\lambda) \le \liminf_{\epsilon\to 0} E^\epsilon(w^\epsilon_{i(\epsilon)}, \Omega)$$

since $w^\epsilon_{i(\epsilon)} - \lambda x \in \mathcal{A}^0_\epsilon$. Moreover, our hypothesis that $u^\epsilon \rightharpoonup \lambda x$ in $H^1(\Omega)$ assures that $|\nabla u^\epsilon - \lambda|^2_{L^2(\Omega\setminus\Omega_0')}$ remains uniformly bounded, and Rellich's lemma assures that $\lim_{\epsilon\to0}|u^\epsilon - \lambda x|^2_{L^2(\Omega\setminus\Omega_0')} = 0$. Therefore we can conclude from (B.9) that

$$|\Omega|\,\overline{W}(\lambda) \le \liminf_{\epsilon\to 0} E^\epsilon(u^\epsilon, \Omega)$$

by taking the $\liminf$ in ϵ, then the limit $\nu\to\infty$, then finally the limit $\delta\to 0$. The proof of lemma 3.1 is now complete.

C Approximating $u \in H^1(\Omega)$ by piecewise affine functions

Near the beginning of section 3.3 we asserted that for any Lipschitz domain Ω and any $u \in H^1(\Omega)$, there is a piecewise linear approximation u_δ of u satisfying (3.23) and (3.24), which we repeat for the reader's convenience:

$$|\tilde{u} - u_\delta|_{H^1(\mathbb{R}^N)} \to 0 \text{ as } \delta \to 0 \quad\text{and}$$
$$|u_\delta|_{L^\infty(\mathbb{R}^N)} + |\nabla u_\delta|_{L^\infty(\mathbb{R}^N)} \le c_u\delta^{-a}\,.$$

Here $\tilde{u}$ is a compactly supported extension of u satisfying $|\tilde{u}|_{H^1(\mathbb{R}^N)} \le C|u|_{H^1(\Omega)}$, c_u is a constant (depending on $|u|_{H^1(\Omega)}$), and a is a positive constant depending only on the spatial dimension N. This appendix provides a detailed justification of that assertion.

Let $u^\eta = \varphi^\eta * \tilde{u}$ be the smooth approximation to $\tilde{u}$ obtained by mollification with $\varphi^\eta = \frac{1}{\eta^N}\varphi\left(\frac{x}{\eta}\right)$, where φ is smooth and compactly supported with integral 1. It is standard that $u^\eta \to u$ in $H^1(\mathbb{R}^N)$, with

$$|u^\eta - \tilde{u}|_{L^2(\mathbb{R}^N)} \le C\eta|\nabla\tilde{u}|_{L^2(\mathbb{R}^N)} \le C\eta|u|_{H^1(\Omega)} \qquad\text{and}\qquad \lim_{\eta\to0}|\nabla u^\eta - \nabla\tilde{u}|_{L^2(\mathbb{R}^N)} = 0\,. \tag{C.1}$$

Moreover, we can bound the L^∞ norms of u^η, ∇u^η and $\nabla\nabla u^\eta$ by

$$|u^\eta|_{L^\infty(\mathbb{R}^N)} \le \frac{M_1}{\eta^{\frac{N}{2}}}|\tilde{u}|_{L^2(\mathbb{R}^N)} \le C\frac{M_1}{\eta^{\frac{N}{2}}}|u|_{H^1(\Omega)} \quad \text{with } M_1 = |\varphi|_{L^2(\mathbb{R}^N)}\,; \tag{C.2}$$

$$|\nabla u^\eta|_{L^\infty(\mathbb{R}^N)} \le \frac{M_1}{\eta^{\frac{N}{2}}}|\nabla\tilde{u}|_{L^2(\mathbb{R}^N)} \le C\frac{M_1}{\eta^{\frac{N}{2}}}|u|_{H^1(\Omega)}\,; \quad \text{and} \tag{C.3}$$

$$|\nabla\nabla u^\eta|_{L^\infty(\mathbb{R}^N)} \le \frac{M_2}{\eta^{\frac{N+2}{2}}}|\nabla\tilde{u}|_{L^2(\mathbb{R}^N)} \le C\frac{M_2}{\eta^{\frac{N+2}{2}}}|u|_{H^1(\Omega)} \quad \text{with } M_2 = |\nabla\varphi|_{L^2(\mathbb{R}^N)}\,. \tag{C.4}$$

The proofs of (C.2)-(C.4) are straightforward; to indicate the method, we provide the details for (C.4). Since $\nabla\nabla u^\eta = (\nabla\varphi^\eta) * \nabla\tilde{u}$, Young's inequality gives

$$|\nabla\nabla u^\eta|_{L^\infty(\mathbb{R}^N)} \le |\nabla\varphi^\eta|_{L^2(\mathbb{R}^N)}|\nabla\tilde{u}|_{L^2(\mathbb{R}^N)} \le C|\nabla\varphi^\eta|_{L^2(\mathbb{R}^N)}|u|_{H^1(\Omega)}$$

and $|\nabla\varphi^\eta|_{L^2(\mathbb{R}^N)}$ can be computed directly:

$$\begin{aligned}\int_{\mathbb{R}^N}|\nabla\varphi^\eta|^2\,dx &= \int_{\mathbb{R}^N}\frac{1}{\eta^{2N+2}}\left|\nabla\varphi\left(\frac{x}{\eta}\right)\right|^2 dx \qquad \text{since } \nabla\varphi^\eta = \frac{1}{\eta^{N+1}}\nabla\varphi\left(\frac{x}{\eta}\right)\\ &= \int_{\mathbb{R}^N}\frac{1}{\eta^{N+2}}|\nabla\varphi(y)|^2\,dy \qquad (x=\eta y)\,.\end{aligned}$$

As already discussed in section 3.3, we choose u_δ to be the piecewise affine interpolant of u^η using a mesh of order δ. We claim that it has the desired properties when η is chosen to depend appropriately on δ; in particular, it is sufficient that

$$\eta = \delta^t \quad \text{with} \quad t = \frac{1}{N+2}\,. \tag{C.5}$$

Indeed, if u_δ^η is the piecewise affine interpolant of u^η then we can estimate $u_\delta^\eta - u^\eta$ using (C.3) and (C.4):

$$|\nabla u_\delta^\eta - \nabla u^\eta|_{L^\infty(\mathbb{R}^N)} \le C'\delta|\nabla\nabla u^\eta|_{L^\infty(\mathbb{R}^N)} \le C'CM_2\frac{\delta}{\eta^{\frac{N+2}{2}}}|u|_{H^1(\Omega)} \quad \text{and} \tag{C.6}$$

$$|u_\delta^\eta - u^\eta|_{L^\infty(\mathbb{R}^N)} \le C'\delta|\nabla u^\eta|_{L^\infty(\mathbb{R}^N)} \le C'CM_1\frac{\delta}{\eta^{\frac{N}{2}}}|u|_{H^1(\Omega)}\,. \tag{C.7}$$

where C' depends on the geometry of the mesh used to define u_δ^η (but not on η or δ). Now,

$$|\tilde{u} - u_\delta^\eta|_{H^1(\mathbb{R}^N)} \le |\tilde{u} - u^\eta|_{H^1(\mathbb{R}^N)} + |u^\eta - u_\delta^\eta|_{H^1(\mathbb{R}^N)}\,. \tag{C.8}$$

The first term on the right tends to 0 as $\eta \to 0$ by (C.1). Since $\tilde{u}$ is compactly supported in $\mathbb{R}^N$, the functions u^η and u_δ^η also have compact support (on sets whose volumes are uniformly bounded for $\delta \le 1$ and $\eta \le 1$). Combining this with (C.6) and (C.7), we see that the second term on the right side of (C.8) tends to zero provided that $\delta/\eta^{(N+2)/2} \to 0$. Thus the choice $\eta = \delta^t$ assures that

$|\tilde{u} - u_\delta^\eta|_{H^1(\mathbb{R}^N)} \to 0$ as $\delta \to 0$ provided that

$$0 < t < 1 \quad \text{and} \quad t < \frac{2}{N+2}\,.$$

For any such choice of t, we get estimates on the L^∞ norms of u_δ and ∇u_δ from (C.2) and (C.3); in particular, the choice $t = 1/(N+2)$ gives

$$|u_\delta|_{L^\infty(\mathbb{R}^N)} + |\nabla u_\delta|_{L^\infty(\mathbb{R}^N)} \leq c_u \delta^{-a} \quad \text{with} \quad a = \frac{N}{2(N+2)} \quad \text{and} \quad c_u = C|u|_{H^1(\Omega)}\,.$$

D The upper and lower bounds for convex 2D polygons

This appendix proves our upper and lower bounds (4.7)-(4.8) for convex 2D polygons. The argument is by induction. The initial step (showing the bounds for triangles) is presented in appendix D.1. The inductive step is then treated in appendix D.2.

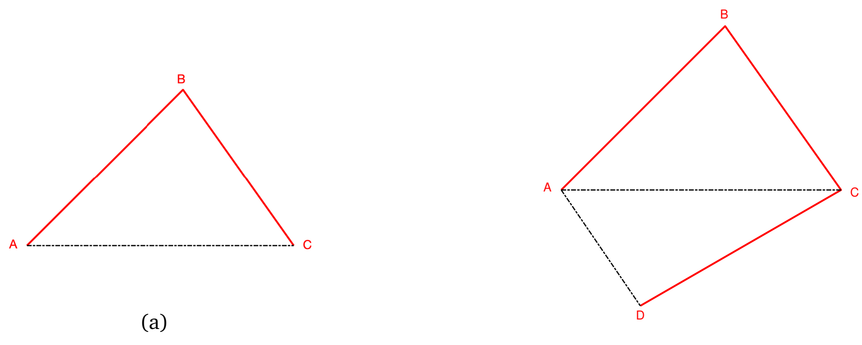


Figure 9: (a) a general triangle; (b) a convex quadrilateral: the red solid edges are counted in the energy $E_{\text{poly}}(u, P_n)$ and the dotted edges indicate the triangular mesh.

D.1 Triangles

When specialized to the triangle ΔABC shown in fig. 9a, our polygon energy reduces to

$$E_{\text{poly}}(u, \Delta ABC) = (|u(A) - u(B)| - |A - B|)^2 + (|u(B) - u(C)| - |B - C|)^2 \tag{D.1}$$

and ∇u becomes the constant matrix characterized by

$$\nabla u = \begin{pmatrix} u(A) - u(B) & u(B) - u(C) \end{pmatrix} \begin{pmatrix} A - B & B - C \end{pmatrix}^{-1} = M_1 M_2^{-1}, \tag{D.2}$$

where we have set $M_1 = \begin{pmatrix} u(A) - u(B) & u(B) - u(C) \end{pmatrix}$ and $M_2 = \begin{pmatrix} A - B & B - C \end{pmatrix}$.

We start with the following elementary upper and lower bounds:

$$E_{\text{poly}}(u, \Delta ABC) \leq |u(A) - u(B)|^2 + |u(B) - u(C)|^2 + |A - B|^2 + |B - C|^2 \quad \text{and} \tag{D.3}$$

$$E_{\text{poly}}(u, \Delta ABC) \geq \frac{1}{2}\Big(|u(A) - u(B)|^2 + |u(B) - u(C)|^2\Big) - 2\Big(|A - B|^2 + |B - C|^2\Big) \tag{D.4}$$

(using for the latter the fact that $(x - y)^2 \geq \frac{1}{2}x^2 - 2y^2$ for any $x, y \in \mathbb{R}$). Our main task is evidently to control the term that's quadratic in u,

$$|u(A) - u(B)|^2 + |u(B) - u(C)|^2 = |M_1|^2 \tag{D.5}$$

in terms of $|\nabla u|^2$. This is easy: we are of course assuming that the triangle is nondegenerate (that is, none of its sides has length zero), so the matrix M_2 is nonsingular. Therefore $|M|$ and $|MM_2^{-1}|$ are both norms on a 2×2 matrix M. Since all norms are equivalent in finite dimensions, there are constants α and β (depending on the geometry of the triangle) such that $|MM_2^{-1}|^2 \geq \alpha|M|^2$ and $|MM_2^{-1}|^2 \leq \beta|M|^2$. Taking $M = M_1$ and integrating, this gives

$$|\nabla u|^2_{L^2(\Delta ABC)} \geq \alpha|M_1|^2|\Delta ABC| \quad \text{and} \quad |\nabla u|^2_{L^2(\Delta ABC)} \leq \beta|M_1|^2|\Delta ABC|\,. \tag{D.6}$$

Combining this with (D.3) – (D.5) gives

$$E_{\text{poly}}(u, \Delta ABC) \leq \frac{1}{\alpha|\Delta ABC|}|\nabla u|^2_{L^2\Delta ABC} + \big(|A - B|^2 + |B - C|^2\big) \quad \text{and}$$
$$E_{\text{poly}}(u, \Delta ABC) \geq \frac{1}{2\beta|\Delta ABC|}|\nabla u|^2_{L^2\Delta ABC} - 2\big(|A - B|^2 + |B - C|^2\big)\,,$$

which clearly imply inequalities of the desired form (4.7)–(4.8).

D.2 The inductive step

The passage from triangles to quadrilaterals provides the main ideas of the inductive step, so we shall start by discussing that. Afterward we'll briefly explain how a similar argument handles the passage from $n - 1$ sided polygons to n sided ones.

Consider the quadrilateral P_4 in fig. 9b, whose energy is

$$E_{\text{poly}}(u, P_4) = E_{AB}(u) + E_{BC}(u) + E_{CD}(u)$$

where $E_{AB}(u) = (|u(A) - u(B)| - |A - B|)^2$, etc. The associated upper bound is easy: we have

$$E_{\text{poly}}(u, P_4) \leq \Big(E_{AB}(u) + E_{BC}(u)\Big) + \Big(E_{AC}(u) + E_{CD}(u)\Big) \tag{D.7}$$

since $E_{AC}(u) \geq 0$. Our result for triangles estimates each of the terms on the right:

$$E_{AB}(u) + E_{BC}(u) \leq c_1(\Delta ABC)\Big(|\nabla u|^2_{L^2(\Delta ABC)} + |\Delta ABC|\Big)],,$$
$$E_{AC}(u) + E_{CD}(u) \leq c_1(\Delta ACD)\Big(|\nabla u|^2_{L^2(\Delta ACD)} + |\Delta ACD|\Big).$$

Combining these estimates gives

$$E_{\text{poly}}(u, P_4) \leq c_{1,\max}\Big(|\nabla u|^2_{L^2(P_4)} + |P_4|\Big)$$

where $c_{1,\max}$ is the larger of the c_1's of the two triangles.

For the lower bound we must work a bit harder. We start with the observation that there exists a constant γ (depending only on the geometry of triangle ABC) such that

$$E_{AC}(u) \leq 3\Big[\gamma + E_{AB}(u) + E_{BC}(u)\Big]. \tag{D.8}$$

The importance of this inequality is that we can bound the energy of one spring in a triangle by the energies of the other two springs. The proof of (D.8) is a straightforward calculation which we briefly postpone. Writing (D.8) in the form $\frac{1}{6}E_{AC} - \frac{\gamma}{2} - \frac{1}{2}(E_{AB} + E_{BC}) \leq 0$, we see that

$$\begin{aligned} E_{\text{poly}}(u, P_4) &\geq E_{AB}(u) + E_{BC}(u) + E_{CD}(u) + \frac{1}{6}E_{AC}(u) - \frac{\gamma}{2} - \frac{1}{2}\big(E_{AB}(u) + E_{BC}(u)\big) \\ &\geq \frac{1}{2}\big(E_{AB}(u) + E_{BC}(u)\big) + \frac{1}{6}\big(E_{AC}(u) + E_{CD}(u)\big) - \frac{\gamma}{2}. \end{aligned} \tag{D.9}$$

Our lower bound for triangles gives

$$E_{AB}(u) + E_{BC}(u) \geq c_2(\Delta ABC)|\nabla u|^2_{L^2(\Delta ABC)} - c_3(\Delta ABC)|\Delta ABC|,$$
$$E_{AC}(u) + E_{CD}(u) \geq c_2(\Delta ACD)|\nabla u|^2_{L^2(\Delta ACD)} - c_3(\Delta ACD)|\Delta ACD|.$$

Combining these estimates leads easily to a result of the desired form

$$E_{\text{poly}}(u, P_4) \geq \tilde{c}_2|\nabla u|^2_{L^2(P_4)} - \tilde{c}_3|P_4|.$$

To finish our discussion of quadrilaterals we now demonstrate (D.8). With the notation

$$\begin{aligned} |u(A) - u(C)| &= a_1 \qquad & |A - C| &= a_2 \\ |u(A) - u(B)| &= b_1 \qquad & |A - B| &= b_2 \\ |u(B) - u(C)| &= c_1 \qquad & |B - C| &= c_2 \end{aligned}$$

the estimate (D.8) is equivalent to

$$(a_1 - a_2)^2 \leq \eta\big[\gamma + (b_1 - b_2)^2 + (c_1 - c_2)^2\big] \tag{D.10}$$

with $\eta = 3$. The triangle inequality implies

$$|b_1 - c_1| \le a_1 < b_1 + c_1, \qquad\qquad |b_2 - c_2| \le a_2 < b_2 + c_2 \,. \tag{D.11}$$

To get (D.10), we begin with the observation that

$$(a_1 - a_2)^2 = a_1^2 - 2a_1a_2 + a_2^2 \le a_1^2 + a_2^2 \le 2(b_1^2 + c_1^2 + b_2^2 + c_2^2)\,. \tag{D.12}$$

The last inequality comes from the triangle inequality (D.11), since for $i = 1, 2$ we have

$$\frac{a_i}{2} \le \frac{b_i + c_i}{2} \le \sqrt{\frac{b_i^2 + c_i^2}{2}} \qquad \Rightarrow \qquad a_i^2 \le 2(b_i^2 + c_i^2)\,.$$

In view of (D.12), (D.10) will hold if

$$2(b_1^2 + c_1^2 + b_2^2 + c_2^2) \le \eta\big[\gamma + (b_1 - b_2)^2 + (c_1 - c_2)^2\big]\,.$$

It is clear that such an estimate should hold for some η and γ, since $b_2 = |A - B|$ and $c_2 = |B - C|$ are constants (independent of u) and both sides are quadratic in $b_1 = |u(A) - u(B)|$ and $c_1 = |u(B) - u(C)|$. In fact the estimate holds with $\eta = 3$ and $\gamma = 6M^2$, if we set $M = \max\{b_2, c_2\}$. To see this, we use the fact that

$$0 \le \frac{1}{2}(b_1 + c_1)^2 - 6M(b_1 + c_1) + 18M^2 \le b_1^2 + c_1^2 + b_2^2 + c_2^2 - 6M(b_1 + c_1) + 18M^2\,.$$

Adding $2(b_1^2 + c_1^2 + b_2^2 + c_2^2)$ to both sides gives

$$\begin{aligned} 2(b_1^2 + c_1^2 + b_2^2 + c_2^2) &\le 3(b_1^2 + c_1^2 + b_2^2 + c_2^2) - 6M(b_1 + c_1) + 18M^2 \\ &\le 3(b_1^2 + c_1^2 + b_2^2 + c_2^2) - 6(b_1b_2 + c_1c_2) + 18M^2 = 3\big[6M^2 + (b_1 - b_2)^2 + (c_1 - c_2)^2\big]\,, \end{aligned}$$

as claimed. This completes our treatment of quadrilaterals.

We now indicate how the same ideas can be used to obtain the upper and lower bounds for convex polygons with n sides, once they are known for polygons with $n-1$ sides. Let P_n have vertices $A_1, \ldots, A_n$ as shown in fig. 4, and recall that

$$E_{\text{poly}}(u, P_n) = E_{A_1A_2}(u) + \ldots E_{A_{n-1}A_n}(u)\,.$$

The upper bound is obtained by observing that

$$E_{\text{poly}}(u, P_n) \le E_{\text{poly}}(u, P')(u) + E_{\text{poly}}(u, P'')$$

where P' is the triangle $A_1A_2A_3$ and P'' is the $n-1$-sided polygon $A_1A_3A_4\ldots A_n$. This is the analogue of (D.7), and by arguing as we did there one obtains the upper bound for $E_{\text{poly}}(u, P_n)$ from the upper bounds for $E_{\text{poly}}(u, P')$ and $E_{\text{poly}}(u, P'')$. The lower bound is obtained by using (D.8) for the triangle

P':

$$E_{\text{poly}}(u, P_n) \geq E_{\text{poly}}(u, P_n) + \frac{1}{6}E_{A_1A_3}(u) - \frac{\gamma}{2} - \frac{1}{2}\big(E_{A_1A_2}(u) + E_{A_2A_3}(u)\big)$$
$$\geq \frac{1}{2}E_{\text{poly}}(u, P') + \frac{1}{6}E_{\text{poly}}(u, P'') - \frac{\gamma}{2}\,.$$

This is the analogue of (D.9), and arguing as we did there gives the lower bound for $E_{\text{poly}}(u, P_n)$ by combining those for $E_{\text{poly}}(u, P')$ and $E_{\text{poly}}(u, P'')$.

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